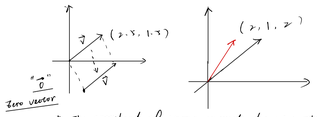
Vector & Dot/Cross Produce

Vector



☆ The arrowhead of vector uniquely determine the vector.
(Since we can always move the vector to the origin)
↳ Use the coordinate of arrowhead to express vector.

Norm of vector

"magnitude" or "length"

$|\vec{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

* $|\vec{0}| = 0$ by definition

Vector addition / Vector Subtraction

$\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}$

$\vec{u} = (x_1, y_1)$

$\vec{v} = (x_2, y_2)$

$\vec{u}+\vec{v} = (x_1+x_2, y_1+y_2)$ ☆ component-wise addition

Vector Scaling

- $k > 0$: $k\vec{u} \longrightarrow$ Same direction. k times length.
- $k = 0$: zero vector.
- $k < 0$: $k\vec{u} \longrightarrow$ opposite direction. $|k\vec{u}| = |k||\vec{u}|$

unit vector

$\vec{u} = \frac{1}{|\vec{v}|} \cdot \vec{v} \Rightarrow |\vec{u}| = 1$

$\vec{v} = (v_1, v_2) = v_1 \cdot \vec{i} + v_2 \cdot \vec{j}$

Dot Product

$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2$

The result is a Scaler $\Rightarrow$ $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

proof:

$$\begin{cases} |\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta \text{ (cosine law)} \\ (\vec{u}-\vec{v})(\vec{u}-\vec{v}) = |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\cdot\vec{v} \end{cases}$$

$\therefore \vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

Orthogonal

$\vec{u}\cdot\vec{v} = 0 \Rightarrow \vec{u}, \vec{v}$ are orthogonal

Orthogonal Projection

$\text{Proj}_{\vec{v}}\vec{u} = \underbrace{|\vec{u}|\cos\theta}_{\text{length}} \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{direction}}$

$= |\vec{u}|\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\frac{\vec{v}}{|\vec{v}|}$

Scaler: $\boxed{\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}}\vec{v}$

$= \frac{\vec{u}\cdot\vec{v}}{|\vec{v}||\vec{v}|}\cdot\vec{v}$

if $\vec{w} = \vec{u} - \text{Proj}_{\vec{v}}\vec{u}$ then $\vec{w} \perp \vec{u}$

Scalar Projection

"the length of the projection"

$\text{Comp}_{\vec{v}}\vec{u} = |\vec{u}|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$

Cross Product

☆ Determinant:

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$$\begin{cases} \vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k} \\ \vec{v} = u_1\vec{i} + v_2\vec{j} + v_3\vec{k} \end{cases}$$

Cross Product:

$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$ This is also a Vector in $\mathbb{R}^3$

☆ $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$

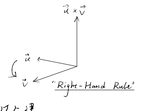
☆ $\vec{u}\times\vec{u} = 0$

"$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix}$" where $\begin{cases} \vec{u} = a\vec{i} + b\vec{j} + c\vec{k} \\ \vec{v} = d\vec{i} + e\vec{j} + f\vec{k} \end{cases}$

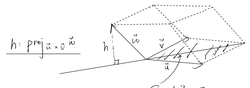
Triple Product

$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$

$\vec{w}\cdot(\vec{u}\times\vec{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$



$$\begin{cases} (\lambda\vec{u})\times(\mu\vec{v}) = (\lambda\mu)(\vec{u}\times\vec{v}) & \text{交換律} \\ \vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w} & \text{分配律} \end{cases}$$



$S = |\vec{u}\times\vec{v}|$

$V = S\cdot h = |\vec{u}\times\vec{v}|\,\text{comp}_{\vec{u}\times\vec{v}}\vec{w}$

$= |\vec{u}\times\vec{v}|\frac{\vec{w}\cdot(\vec{u}\times\vec{v})}{|\vec{u}\times\vec{v}|}$

$= \vec{w}\cdot(\vec{u}\times\vec{v})$ Triple Product!

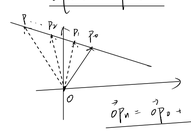
$= \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$$\boxed{\begin{aligned} &|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta. \\ &|\vec{u}||\vec{v}|\sin\theta = |\vec{u}||\vec{v}|\sqrt{1-\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)^2} \\ &\text{The "square" of parallelogram} = \sqrt{|\vec{u}|^2|\vec{v}|^2 - \frac{(\vec{u}\cdot\vec{v})^2}{|\vec{u}|^2|\vec{v}|^2}} \\ &= \sqrt{|\vec{u}|^2|\vec{v}|^2 - (\vec{u}\cdot\vec{v})^2} \\ &= \sqrt{(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2} \\ &= \sqrt{(u_1v_2-u_2v_1)^2 + (u_1v_3-u_3v_1)^2 + (u_2v_3-u_3v_2)^2} \end{aligned}}$$

$S = |\vec{u}||\vec{v}|\sin\theta \Rightarrow = |\vec{u}\times\vec{v}|$

Line & Plane

Equation of Line



$\vec{OP_n} = \vec{OP_0} + \vec{P_0P_n}$

$\vec{P_0P} = t\vec{v} = t\langle a, b, c\rangle$

$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$

equation of line

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} \xrightarrow{\text{symmetric form}} \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

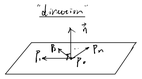
Remark:

The equation is not unique. i.e. We can always find different $P(x_0, y_0, z_0)$ in the same line.

Equation of Plane

Plane $\longrightarrow$ Normal Vector

"direction"



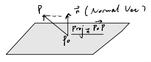
"$\vec{P_0P}\cdot\vec{n} = 0$"

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$ Point-normal form

$ax + by + cz + d = 0$ General Form

$ax_0 + by_0 + cz_0$

Distance Problem



$d = |\text{Proj}_{\vec{n}}\vec{P_0P}| = \frac{|\vec{n}\cdot\vec{P_0P}|}{|\vec{n}|}$

$P(x', y', z')$, $ax + by + cz + d = 0$

☆ $d = \frac{|ax' + by' + cz' + d|}{\sqrt{a^2+b^2+c^2}}$

Vector-Valued function

$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$

e.g. a line: $\vec{r}(t) = \langle at+x_0, mt+b, z_0+ct\rangle$

Remarks:

1. Same Curve can be defined by different V-V function.
2. $t \in [a, b]$, then positive direction: $a \to b$.

Limit and Continuity

$\underbrace{\lim_{t\to a}\vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle}_{\text{Exist iff}}$ Exist.

Continuity

$\vec{r}(t)$ is Continuous at $t = a$.

if $\lim_{t\to a}\vec{r}(t) = \vec{r}(a)$

☆ i.e. if $f(t), g(t), h(t)$ are Continuous at $t = a$

Calculus of Vector-Valued Functions

$\vec{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$

$\int\vec{r}'(t)\,dt = \vec{r}(t) + C$

Motion in Space

Trajectory $\vec{r}(t)$

Speed $v(t) = \vec{r}'(t)$

acceleration $a(t) = \vec{v}'(t) = \vec{r}''(t)$

$\vec{r}(t) = \langle x, y, z\rangle$

$\vec{v}(t) = \vec{r}'(t) = \langle g\cos\theta\cos\alpha, g\cos\theta\sin\alpha, g\sin\theta\rangle$

$\vec{v}(t) = \vec{v}(0) + \int_0^t\vec{a}(t)\,dt$

- $\vec{v}(t) = \vec{v}(0) + \int_0^t\langle 0, 0, -g\rangle\,dt$
- $\vec{v}(t) = \langle v_1, v_2, v_3\rangle + \langle 0, 0, -gt\rangle\Big|_0^t$
- $= \langle g\cos\theta\cos\alpha, g\cos\theta\sin\alpha, g\sin\theta - gt\rangle$

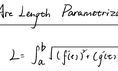
$\vec{r}(t) = \vec{r}(0) + \int_0^t\vec{v}(t)\,dt$

- $\vec{r}(t) = \langle(g\cos\theta\cos\alpha)t, (g\cos\theta\sin\alpha)t, g\sin\theta\cdot t - \frac{1}{2}gt^2\rangle$
- $\vec{r}(t) = \langle(g\cos\theta\cos\alpha)t, (g\cos\theta\sin\alpha)t, g\sin\theta\cdot t - \frac{1}{2}gt^2\rangle$

$g\sin\theta\,T - \frac{1}{2}gT^2 = 0$ where T = the time when ball hits the ground again

$T = \frac{2g\sin\theta}{g}$

Range R



$R = |\vec{r}(T)|$

Length and Arc Length Parametrization of Curves

arc length $L = \int_a^b\sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} = \int_a^b|\vec{r}'(t)|\,dt$.

Reparametrization:

$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$

$t = \phi(s)$, where ϕ: Smooth strictly increasing function

$\vec{r}(s) = \langle f(\phi(s)), g(\phi(s)), h(\phi(s))\rangle$ with $\phi^{-1}(a) < s < \phi^{-1}(b)$

"reparametrization": actually parametrizes the Same curve

Reparametrization by arc length:

$t = \phi(s)$ where s is the arc length from $\vec{r}(a)$ to $\vec{r}(t)$

$g = \phi^{-1}(t) = \int_a^t\sqrt{|\vec{r}'(u)|^2}\,du$ = "a function with parameter t"

$\vec{r}(t) = \langle f(\phi(s)), g(\phi(s)), h(\phi(s))\rangle$

Curvature of Curves

Curvature of curves.

$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$

unit tangent vector

$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ "A direction"

Curvature: How "Sharp" a curve is.

i.e. How much the curve direction changes over a small distance

$K(s) = \left|\frac{d\vec{T}}{ds}\right|$ $\vec{T}$: unit tangent vector.

S: arc length $\Rightarrow S = \int_a^b\sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,dt$

$K = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}\right| = \frac{1}{|\vec{r}'|}\cdot\left|\frac{d\vec{T}}{dt}\right|$

$\Rightarrow K = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$

Remarks:

1. $K = 0$ anywhere in a line
   $\vec{v} = \vec{r}'(t) = \langle a, b, c\rangle$, i.e. a constant
   $\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle a, b, c\rangle}{\sqrt{a^2+b^2+c^2}}$. (a constant) $\Rightarrow \vec{T}' = 0$
   $K = \left|\frac{\vec{T}'}{\vec{v}}\right| = 0$
2. $K = \frac{1}{R}$ on the circle w/ radius R

An alternative formula

Let $\vec{a}(t) = \vec{v}'(t)$. ($\vec{v}(t) = \vec{r}'(t)$)

$\vec{a} = \frac{d}{dt}\vec{v}(t) = \frac{d}{dt}(|\vec{v}|\cdot\vec{T}) = |\vec{v}|\cdot\frac{d}{dt}\vec{T} + \frac{d}{dt}|\vec{v}|\cdot\vec{T}$

$\vec{v}\times\vec{a} = (|\vec{v}|\cdot\vec{T})\times\left(|\vec{v}|\cdot\frac{d}{dt}\vec{T} + \frac{d}{dt}|\vec{v}|\cdot\vec{T}\right)$

$= (|\vec{v}|\cdot\vec{T})\times(|\vec{v}|\cdot\frac{d}{dt}\vec{T})$ // $\vec{T}\times\vec{T} = \vec{0}$

$= |\vec{v}|^2\cdot\vec{T}\times\frac{d}{dt}\vec{T}$

$|\vec{v}\times\vec{a}| = \left||\vec{v}|^2\cdot\vec{T}\times\frac{d}{dt}\vec{T}\right| = |\vec{v}|^2\boxed{|\vec{T}|\cdot\left|\frac{d}{dt}\vec{T}\right|}\sin\theta$

$= |\vec{v}|^2\cdot\left|\frac{d}{dt}\vec{T}\right|$

$K = \frac{1}{|\vec{v}|}\cdot\left|\frac{d}{dt}\vec{T}\right|$

$= \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}$

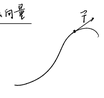
Since $\vec{T}\cdot\vec{T} = 1$

$\frac{d}{dt}(\vec{T}\cdot\vec{T}) = 0$

$\Rightarrow \vec{T}\left(\frac{d}{dt}(\vec{T})\right) = 0 \Rightarrow \vec{T}$ and $\vec{T}'$ are orthogonal

Principal unit normal vector

$\vec{N}(t) = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} = \frac{\frac{d\vec{T}}{ds}}{\left|\frac{d\vec{T}}{ds}\right|}$ 與法向量



Since $K = \left|\frac{d\vec{T}}{ds}\right|$

$\frac{d\vec{T}}{ds} = K\vec{N}$

also, $\vec{N}$, $\vec{T}$ are orthogonal since $\vec{T}\cdot\frac{d\vec{T}}{ds} = 0$

Osculating Plane:

Plane contains $\vec{T}$ and $\vec{N}$

Osculating Circle:

center: $C \Rightarrow \vec{PC}$ is in same direction with $\vec{N}$

radius $R = \frac{1}{K}$

This circle is the best approximate of the curve at P