## Vector Field

$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$

## Gradient Field

$\vec{F}(x,y) = \nabla \phi(x,y) = \langle \phi_x(x,y), \phi_y(x,y) \rangle$

where $\phi(x,y)$ is called potential function

remark: $\phi(x,y)$ is potential function $\Rightarrow$ $\phi(x,y)+C$ is potential function

Def: $\vec{F}$ is conservative field if $\exists \phi$. $\vec{F} = \nabla\phi$

e.x. Example: Let $\vec{F}(x,y,z) = \frac{\vec{r}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ $(x,y,z)$ be a radial vector field in 3D space, where $C$ is a constant. Show that it is a conservative field.

(ex $\phi = \frac{-C}{\sqrt{x^2+y^2+z^2}}$, $\nabla\phi = \frac{C}{(x^2+y^2+z^2)^{\frac{3}{2}}} \langle x, y, z \rangle$

$\Rightarrow$ Conservative

Exercise:

1. Show that $\vec{F} = \langle y, x \rangle$ is a conservative field. (Hint: Guess the potential function.)
2. Show that $\vec{G} = \langle -y, x \rangle$ is not a conservative field. (Hint: Suppose $\phi$ is a potential function of $\vec{G}$ i.e. $\vec{G} = \nabla\phi$. Consider $\frac{\partial^2 \phi}{\partial x \partial y}$ and $\frac{\partial^2 \phi}{\partial y \partial x}$.)

(1) let $\phi = xy$ $\nabla\phi = \langle y, x \rangle \Rightarrow$ conservative

(2) Suppose $\vec{G} = \nabla\phi$

$\frac{\partial \phi}{\partial x} = -y \Rightarrow \frac{\partial^2 \phi}{\partial x \partial y} = -1$

$\frac{\partial \phi}{\partial y} = x \Rightarrow \frac{\partial^2 \phi}{\partial y \partial x} = 1$

$\Rightarrow \frac{\partial^2 \phi}{\partial x \partial y} \neq \frac{\partial^2 \phi}{\partial y \partial x}$ and they're supposed to be equal

$\Rightarrow$ not conservative.

## Line integral

$f(x,y) : \mathbb{R}^2$

$C: \vec{r}(t) = \langle x(t), y(t) \rangle$ (arc length parametrization)

$\int_C f\, ds = \int_0^L f(x(t), y(t))\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x(s_k^*), y(s_k^*))\, \Delta s_k$

$\vec{r}(t) = \langle x(t), y(t) \rangle$ $a \le t \le b$.

arc length: $s(t) = \int_a^t \sqrt{x'(t)^2 + y'(t)^2}\, dt$.

$\Rightarrow \frac{ds}{dt} = |\vec{r}'(t)|$

$\int_C f\, ds = \int_C f(x(t), y(t))\, |\vec{r}'(t)|\, dt$

## Line integral of a vector field

$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$

$C: r(t) = \langle x(t), y(t) \rangle$

$\vec{T}$ be unit tangent vector at each point of $\vec{C}$

Line integral of $\vec{F}$: $\int_C \vec{F} \cdot \vec{T}\, ds$

$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ $ds = |r'(t)|\, dt$

$\Rightarrow \int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \langle f(x,y), g(x,y) \rangle \cdot \vec{r}'(t)\, dt$.

$= \int_a^b f(x(t), y(t)) \cdot x'(t) + g(x(t), y(t))\, y'(t)\, dt$.

$= \int_C f\, dx + g\, dy$

Example: Evaluate $\int_C yx^2\, dx + 2\sin(\pi y)\, dy$, where $C$ is the line segment from $(0,2)$ to $(1,4)$.

$C: \vec{r}(t) = \langle t, 2+2t \rangle$ where $0 \le t \le 1$

$\int_C yx^2\, dx + 2\sin(\pi y)\, dy$

$= \int_0^1 (2+2t)\, t^2\, dt + \int_0^1 4\sin(2\pi + 2\pi t)\, dt$

$= \frac{2}{3}t^3 + \frac{1}{2}t^4 - \frac{4}{2\pi}\cos 2\pi t \Big|_0^1$

$= \frac{7}{6}$

Exercise:

1. Evaluate $\int_C 2xy\, dx + (x^2+y^2)\, dy$, where $C: \vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \le t \le \frac{\pi}{2}$.
2. Evaluate $\int_D y\, dx + x\, dy + z\, dz$, where $D: \vec{r}(t) = \langle e^{-2t}, e^t, t^2 \rangle$ for $0 \le t \le 1$.

1. $\int_C 2xy\, dx + (x^2+y^2)\, dy$

$= \int_0^{\frac{\pi}{2}} 2\cos t \sin t \cdot (-\sin t) + \cos t\, dt$.

$= -2\int_0^{\frac{\pi}{2}} \sin^2 t\, d\sin t + \sin t$.

$= -\frac{2}{3}\sin^3 t + \sin t \Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{3}$

2. $= \int_0^1 e^{-t} \cdot 2e^{-2t} + e^{-2t} e^t + t^2 \cdot 2t\, dt$.

$= \int_0^1 -e^{-t} + 2t^3\, dt$.

$= e^{-t} + \frac{1}{2}t^4 \Big|_0^1 = e^{-1} - \frac{1}{2}$

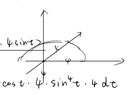
e.x. Example: Find the surface area of the cylinder $x^2+y^2=1$ between xy plane and the parabolic cylinder $z = 1 - x^2$.

$C: x^2+y^2=1$. $\vec{r}(t) = \langle \cos t, \sin t \rangle$ $f(x,y) = 1 - x^2 = 1 - \cos^2 t$.

$\int_C f\, ds = \int_0^{2\pi} (1-\cos^2 t)\, 1\, dt = \int_0^{2\pi} \sin^2 t\, dt = \int_0^{2\pi} \frac{1-\cos 2t}{2}\, dt$.

$= \frac{1}{2}t - \frac{1}{4}\sin 2t \Big|_0^{2\pi} = \pi$

e.x. Exercise: Let $f(x,y) = xy^3$. Evaluate $\int f\, ds$, where $C$ is the right half of the circle centered at the origin with radius 4 traced out in anti-clockwise direction.



$C: r(t) = \langle 4\cos t, 4\sin t \rangle$

$\int_C f\, ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos t \cdot 4^3 \sin^4 t \cdot 4\, dt$

$= 4^6 \int_0^{\pi} \cos t \sin^4 t\, dt$

$= \frac{4^6}{5} \sin^5 t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{2 \cdot 4^6}{5}$

e.x. Example: Evaluate $\int_L (xy + 2z)\, ds$, where $L$ is the line segment from $(1,0,0)$ to $(0,1,1)$.

$L: r(t) = \langle 1-t, t, t \rangle$ where $t \in [0,1]$

$\int_L (xy + 2z)\, ds = \int_0^1 (t(1-t) + 2t) \cdot \sqrt{3}\, dt$.

$= \sqrt{3}\int_0^1 -t^2 + 3t\, dt$.

$= \sqrt{3}\left(-\frac{1}{3}t^3 + \frac{3}{2}t^2\right)\Big|_0^1$

$= \sqrt{3}\,\frac{7}{6}$

Exercise: Evaluate $\int_H xyz\, ds$, where $H$ is the helix given by $\vec{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$ for $0 \le t \le 4\pi$.

$\int_H xyz\, ds = \int_0^{4\pi} \cos t \sin t \cdot 3t \cdot \sqrt{10}\, dt$

$= 3\sqrt{10}\int_0^{4\pi} t \cdot \cos t \sin t\, dt$.

$= \frac{3\sqrt{10}}{2}\int_0^{4\pi} t \sin 2t\, dt$.

$= \frac{-3\sqrt{10}}{4}\int_0^{4\pi} t\, d\cos 2t$

$= \frac{-3\sqrt{10}}{4} \cdot \left( t\cos 2t - \int_0^{4\pi} \cos 2t\, dt \right)$

$= \frac{-3\sqrt{10}}{4}\left( t\cos 2t - \frac{1}{2}\sin 2t \right)\Big|_0^{4\pi} = 0$.

$\pi$

$4\pi$

$\frac{3\pi}{4}$

e.x. Exercise: Determine whether the vector field $\vec{F}(x,y) = \langle e^x \cos y, -e^x \sin y \rangle$ is conservative on $\mathbb{R}^2$. Find its potential function if it is conservative.

Type your answer here...

$\frac{\partial f}{\partial y} = -e^x \sin y$ $\frac{\partial g}{\partial x} = -e^x \sin y \Rightarrow$ Conservative.

$\phi = \int f\, dx = e^x \cos y + k(y)$

$g = \frac{\partial \phi}{\partial y} = -e^x \sin y + k'(y) \Rightarrow k'(y) = 0$. $k(y) = C$.

$\Rightarrow \phi(x,y) = e^x \cos y + C$

Theorem **(Fundamental theorem for line integrals)** Let $\vec{F}$ be a continuous vector field on an open connected region $R$ in $\mathbb{R}^2$ (or $\mathbb{R}^3$). is conservative on i.e. there exists a potential function $\phi$ such that $\vec{F} = \nabla\phi$ if and only if

$$\int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A)$$

for all points $A$ and $B$ in $R$ and all smooth oriented curves $C$ from $A$ to $B$.

Theorem: Let $\vec{F}$ be a 2D vector field defined on a connected region $R$. The following statements are equivalent:

1. $\vec{F}$ is conservative on $R$.
2. $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve $C$ in $D$.
3. $\int_C \vec{F} \cdot d\vec{r}$ is path independent i.e. the line integral has the same value for any two points $A$ and $B$ and any curve from $A$ to $B$

Example: How much work is required to move an object in the force field $\vec{F}(x,y,z) = \langle yz, xy, xz \rangle$ along the path $\vec{r}(t) = \langle t^2, t, t^4 \rangle$ for $0 \le t \le 1$?

$W = \int_0^1 t^5 \cdot 2t + t^3 + t^6 \cdot 4t^3\, dt$.

$= \int_0^1 2t^6 + t^3 + 4t^9\, dt$

$= \frac{2}{7}t^7 + \frac{1}{4}t^4 + \frac{2}{5}t^{10} \Big|_0^1$

$= \frac{131}{140}$

Example: Gravitational force between point masses obeys inverse square laws - the force acts along the line joining the centers of two masses and they vary as $\frac{1}{r^2}$, where $r$ is the distance between the centers. The gravitational force of attraction is given by the vector field

$$\vec{F}(x,y,z) = \frac{k}{(x^2+y^2+z^2)^{\frac{3}{2}}}\langle x, y, z \rangle$$

where $k < 0$ is a physical constant. Find the work done in moving a mass from point $(1,1,1)$ to $(a,a,a)$ along the line segment joining them, where $a > 1$. Moreover, find the work done as $a \to \infty$.

$C: \vec{r}(t) = \langle t, t, t \rangle$ where $t \in [1, a]$.

$\int_C \vec{F} \cdot d\vec{r} = \int_1^a \frac{3kt}{(3t^2)^{\frac{3}{2}}}\, dt$.

$= \int_1^a \frac{3k}{3\sqrt{3} \cdot t^2}\, dt$.

$= \frac{k}{\sqrt{3}} \cdot -\frac{1}{t}\Big|_1^a$

$= \frac{k}{\sqrt{3}}\left(1 - \frac{1}{a}\right)$

when $a \to \infty$. $W = \frac{k}{\sqrt{3}}$

Answer: We already knew that an inverse square field is a conservative field and its potential function (energy) is

$$\phi(x,y,z) = -\frac{k}{(x^2+y^2+z^2)^{\frac{1}{2}}}$$

Therefore, the work done $W_c$ along any path $C$ from $(1,1,1)$ to $(a,a,a)$ is

$$W_c = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \phi(a,a,a) - \phi(1,1,1) = -\frac{k}{a\sqrt{3}} + \frac{k}{\sqrt{3}} = \frac{k}{\sqrt{3}}\left(1 - \frac{1}{a}\right)$$

As $a \to \infty$, $W_c \to \frac{k}{\sqrt{3}}$.

## Green's Theorem

$\oint_C \vec{F} \cdot d\vec{r} = \int_C f\, dx + g\, dy = \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$.

Proof:

(a) $\int_C f(x,y)\, dx = -\iint_R \left( \frac{\partial f}{\partial y} \right) dA$.

$R$: type I region. : $C_1: y = g_1(x)$ $C_2: y = g_2(x)$

$C_1$: $\vec{r}_1(t) = \langle t, g_1(t) \rangle$

$-C_2$: $\vec{r}_2(t) = \langle t, g_2(t) \rangle$

$\int_C f\, dx = \int_{C_1} f\, dx + \int_{-C_2} f\, dx$

$= \int_{C_1} f\, dx - \int_{C_2} f\, dx$.

$= \int_a^b f(t, g_1(t)) - f(t, g_2(t))\, dt$.

$= -\int_a^b \int_{g_1(t)}^{g_2(t)} \frac{\partial f}{\partial y}\, dy\, dt$.

$= -\iint_R \frac{\partial f}{\partial y}\, dA$

(b) $\int_C g\, dy = \int_{-C_1} g\, dy + \int_{C_2} g\, dy$

$= \iint_R \frac{\partial g}{\partial x}\, dA$

$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \int_C f\, dx + g\, dy = \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$.

Example: We revisit the earlier example - evaluate $\int_C x^2 y\, dx + x\, dy$, where $C$ is the loop along the triangle from $(0,0)$ to $(1,0)$ to $(1,2)$ and back to $(0,0)$. Here we use Green's theorem to find the line integral.

Counter-Clockwise!

$\int_0^1 \int_0^{2x} (1 - x^2)\, dy\, dx = \frac{1}{3}$

Finding area of a region

We can use Green's theorem to find the area of a simply connected region $R$ using the vector field $\vec{F}(x,y) = \langle -y, x \rangle$. Suppose we are given the parametrization of the boundary of $R$ (oriented counterclockwise). Then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R (1 - (-1))\, dA = 2\iint_R 1\, dA = 2 \times \text{Area of } R$$

Remark: Other vector fields like $\vec{F}(x,y) = \langle y, 0 \rangle$ also works. The mechanism of a useful tool called plainmeter, which is used for measuring the area of an arbitrary shape by tracing its boundary, is based on Green's theorem.

Example: Let $C$ be an ellipse parametrized by $\vec{r}(t) = \langle a\cos(t), b\sin(t) \rangle$ for $0 \le t \le 2\pi$. Use Green's theorem to find the area of the region $R$ enclosed by $C$.

$S = \iint_R 1\, dA$

$2\iint_R 1\, dA = \int_C f\, dx + g\, dy$ where $f(x,y) = -y$ $g(x,y) = x$

$= \int_0^{2\pi} ab\sin^2 t + ab\cos^2 t\, dt$.

$= (abt)\Big|_0^{2\pi} = 2ab\pi$.

$\therefore S = \pi ab$

Exercise: Use Green's theorem to find the area enclosed by the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, where $a > 0$. The applet below shows the astroid.

(Hint: You can use the parametrization $\vec{r}(t) = \langle a\cos^3(t), a\sin^3(t) \rangle$ for $0 \le t \le 2\pi$.)

Type your answer here...

$C: \vec{r}(t) = \langle a\cos^3(t), a\sin^3(t) \rangle$

$S = \frac{1}{2}\int_C (-y)\, dx + x\, dy$

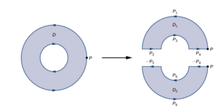
$= \frac{1}{2}\int_0^{2\pi} a\sin^3(t) \cdot 3a\cos^2(t) \cdot \sin t + a\cos^3 t \cdot 3a\sin^2(t)\cos(t)\, dt$.

$= \frac{1}{2}\int_0^{2\pi} 3a^2 \sin^2(t)\cos^2 t\, dt$.

$= \frac{3a^2}{2}\int_0^{2\pi} \sin^2(t) - \sin^4 t\, dt$.

$= \frac{3a^2}{2} \cdot \left( \pi - \frac{3}{4}\pi \right) = \frac{3\pi a^2}{8}$

Let $D$ be an annulus, as shown in the image below




The boundary of the annulus consists of two circles - the outer one is oriented counterclockwise and the inner one is oriented clockwise. Then we make two "cuts" at a point $P$ and divide $D$ into two simply connected regions $D_1$ and $D_2$, enclosed by the curves as shown above. By Green's theorem, we have

$$\iint_{D_1} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_{D_1} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA + \iint_{D_2} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

$$= \int_{P_1P_2P_3P_4P_1} f\, dx + g\, dy + \int_{P_6P_5-P_3P_2P_1P_6-P_1} f\, dx + g\, dy$$

$$= \int_{P_1} f\, dx + g\, dy + \int_{P_2} f\, dx + g\, dy + \int_{P_3} f\, dx + g\, dy + \int_{P_4} f\, dx + g\, dy$$

$$+ \int_{P_5} f\, dx + g\, dy + \int_{-P_2} f\, dx + g\, dy + \int_{P_6} f\, dx + g\, dy + \int_{-P_1} f\, dx + g\, dy$$

$$= \int_{P_1P_2P_3} f\, dx + g\, dy + \int_{P_6P_5P_1} f\, dx + g\, dy$$




$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle$

$\vec{F}$ is conservative vector field iff $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$

e.x. Example: Suppose $\vec{F}(x,y) = \langle y + x, y - x \rangle$ on $\mathbb{R}^2$. Determine whether $\vec{F}$ is conservative on $\mathbb{R}^2$.

$\frac{\partial f}{\partial y} = 1$ $\frac{\partial g}{\partial x} = -1 \Rightarrow$ not conservative.

Example: Suppose $\vec{F}(x,y) = \langle 2xy^3, 1 + 3x^2y^2 \rangle$ on $\mathbb{R}^2$. Show that $\vec{F}$ is conservative on $\mathbb{R}^2$ and find its potential function $\phi$.

Answer:

$\mathbb{R}^2$ is obviously a connected and simply connected region.

$\frac{\partial(2xy^3)}{\partial y} = 6xy^2$ and $\frac{\partial(1+3x^2y^2)}{\partial x} = 6xy^2$.

They are equal. Hence $\vec{F}$ is conservative on $\mathbb{R}^2$.

Let $\phi$ be its potential function. Then we have

$\nabla\phi = \left\langle \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \right\rangle = \langle 2xy^3, 1 + 3x^2y^2 \rangle$

Hence $\frac{\partial\phi}{\partial x} = 2xy^3$, which implies that

$\phi(x,y) = \int 2xy^3\, dx = x^2y^3 + k(y)$, where $k(y)$ is a function of $y$.

Since $\frac{\partial\phi}{\partial y} = 1 + 3x^2y^2$, using the above result, we have

$\frac{\partial\phi}{\partial y} = 3x^2y^2 + k'(y) = 1 + 3x^2y^2$

$\implies k'(y) = 1 \implies k(y) = y + C$, where $C$ is an arbitrary constant.

Hence $\phi(x,y) = x^2y^3 + y + C$.