Double integral over rectangular region

As $n \to \infty$, $\Delta A_i \to 0$ and if limit of the above "Riemann sum" exists for all partitions of R and for all choices of (x_i^*, y_i^*) within those partitions, then the limit is the volume under the graph over R and is called the **double integral** of f over R, denoted as follows:

$$\iint_R f(x,y)\, dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_i^*, y_i^*) \Delta A_i$$

Fubini's theorem: Suppose $z = f(x,y)$ on $R = [a,b] \times [c,d]$. Then

$$\iint_R f(x,y)\, dA = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

e.x. $\iint_R (4 - x - y)\, dA$, where $R = [0,1] \times [0,2]$.

$$\iint_R (4-x-y)\, dA = \int_0^1 \int_0^2 (4-x-y)\, dy\, dx.$$
$$= \int_0^1 4x - \tfrac{1}{2}x^2 - y \Big|_0^2 dy$$
$$= \int_0^1 (7 - 2 - y)\, dy = \tfrac{7}{2}y - \tfrac{1}{2}y^2 \Big|_0^1 = \tfrac{7}{2} - \tfrac{1}{2} = 5.$$

e.x. $\iint_R (4 + 8x^2 y)\, dA$, where $R = [-1,1] \times [0,2]$.

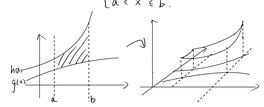
$$\iint_R (4 + 8x^2 y)\, dA = \int_0^2 \int_{-1}^1 (4 + 8x^2 y)\, dx\, dy$$
$$= \int_0^2 \left[4x + 3y \cdot x^3 \right] \Big|_{-1}^1 dy$$
$$= \int_0^2 (8 + 6y)\, dy$$
$$= 8y + 3y^2 \Big|_0^2 = 16 + 12 = 32.$$

e.x. $\iint_R y e^{xy}\, dA$, where $R = [0,1] \times [0, \ln 2]$

$$\iint_R y e^{xy}\, dA = \int_0^{\ln 2} \int_0^1 y e^{xy}\, dx\, dy$$
$$= \int_0^{\ln 2} \frac{y}{y} \cdot e^{xy} \Big|_0^1 dy$$
$$= \int_0^{\ln 2} e^y - 1\, dy$$
$$= e^y - y \Big|_0^{\ln 2} = 2 - \ln 2 - 1 = 1 - \ln 2.$$

Double integral over general regions

Type I region $\begin{cases} \text{upper boundary: } y = h(x) \\ \text{lower boundary: } y = g(x) \\ a \le x \le b. \end{cases}$



$$\iint_D f(x,y)\, dA = \int_a^b \int_{h(x)}^{g(x)} f(x,y)\, dy\, dx.$$

e.x. $f(x,y) = 3xy$ $\iint_D f(x,y)\, dA$, D: region bounded by $y = 3x^2$ and $y = 16 - x^2$

$3x^2 = 16 - x^2$. $x = \pm 2$.

$$\iint_D f(x,y)\, dA = \int_{-2}^{2} \int_{h(x)}^{g(x)} 3xy\, dy\, dx$$
$$= \int_{-2}^{2} xy^2 \Big|_{3x^2}^{16 - x^2} dx$$
$$= \int_{-2}^{2} x \cdot (16 - x^2)^2 - x(3x^2)^2\, dx.$$
$$= \int_{-2}^{2} 3x[x - 32x^2 + x^5 - 9x^3]\, dx.$$
$$= 12x^2 - 8x^4 + \tfrac{1}{6}x^6 - \tfrac{9}{6}x^6 \Big|_{-2}^{2}$$
$$= 0$$

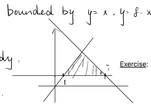
Type II region: $\begin{cases} \text{left boundary } x = p(y) \\ \text{right boundary } x = q(y) \\ a \le y \le b. \end{cases}$

$$\iint_E f(x,y)\, dA = \int_a^b \int_{p(y)}^{q(y)} f(x,y)\, dx\, dy.$$

e.x. $f(x,y) = x \sin y$. $\iint_E f(x,y)\, dA$. E is region bounded by $x = 0$, $x = \cos(y)$, $0 \le y \le \frac{\pi}{2}$.

$$\iint_E f(x,y)\, dA = \int_0^{\frac{\pi}{2}} \int_0^{\cos y} x \sin y\, dx\, dy$$
$$= \int_0^{\frac{\pi}{2}} \left(\tfrac{1}{2} \sin y\, x^2 \right) \Big|_0^{\cos y} dy$$
$$= \int_0^{\frac{\pi}{2}} \tfrac{1}{2} \cdot \sin y \cdot \cos^2 y\, dy$$
$$= -\int_0^{\frac{\pi}{2}} \tfrac{1}{2} \cos^2 y\, d\cos y$$
$$= -\tfrac{1}{6} \cos^3 y \Big|_0^{\frac{\pi}{2}} = -\tfrac{1}{6} \cdot \tfrac{1}{8} + \tfrac{1}{6} \cdot 1 = \tfrac{1}{6} - \tfrac{1}{48} = \tfrac{7}{48}.$$

e.x $f(x,y) = 2x + \frac{1}{y}$, $\iint_E f(x,y)\, dA$. E is region bounded by $y = x$, $y = 8 - x$, $y = 1$



$$\iint_E f(x,y)\, dA = \int_1^7 \int_y^{8-y} \left(2x + \tfrac{1}{y}\right) dx\, dy.$$
$$= \int_1^7 \left(2x + \tfrac{1}{y} x\right) \Big|_y^{8-y} dy$$
$$= \int_1^7 (2 + \tfrac{1}{y})(8 - 2y)\, dy$$
$$= \int_1^7 \left(16 - 4y + 8\tfrac{1}{y}\right) dy$$
$$= 16y - 2y^2 + 8 \ln|y| \Big|_1^7$$
$$= 98 - 98 + 8 \ln 7 - 16 + 2$$
$$= 8 \ln 7 - 12.$$

Exercise: Consider $\iint_T e^{x^2}\, dA$, where T is the triangular region bounded by $y = 2x$, $y = 0$, and $x = 1$.

(a) Express the double integral as an iterated integral by regarding T as a Type II region.
(b) Express the double integral as an iterated integral by regarding T as a Type I region.
(c) Which of the above is easier to compute? Find the value of the double integral.

Type your answer here...

(a) $\iint_T e^{x^2}\, dA = \int_0^2 \int_{2x}^1 e^{x^2}\, dx\, dy$.

(b) $\iint_T e^{x^2}\, dA = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx$.

(c). for (a): $\iint_T e^{x^2}\, dA = \int_0^2 \int_{2x}^1 e^{x^2}\, dx\, dy$.

since $\int e^{x^2} dx$ is hard to compute.

this one is harder.

for (b). $\iint_T e^{x^2}\, dA = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx$.

$$= \int_0^1 y e^{x^2} \Big|_0^{2x} dx$$
$$= \int_0^1 2x e^{x^2}\, dx.$$
$$= e^{x^2} \Big|_0^1 = e - 1.$$

Solid Between Two Surfaces

Solid between two surfaces

Given two functions $f(x,y)$ and $g(x,y)$ such that $f(x,y) \ge g(x,y)$ on a region S. We can compute the volume V between the graphs of $z = f(x,y)$ and $z = g(x,y)$ over the region S by the following double integral:

$$V = \iint_S (f(x,y) - g(x,y))\, dA$$

e.x. Find the Volume of the solid Bounded by $z = x^2 + y^2$ from Below and $z = 8 - (x^2 + y^2)$ from above

$\Rightarrow$ curve of intersection. $x^2 + y^2 = 4$.

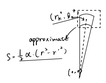
$\therefore V = \iint_S (f(x,y) - g(x,y))\, dA$.

$$= 4 \int_0^2 \int_0^{\sqrt{4-x^2}} 8 - 2(x^2 + y^2)\, dy\, dx.$$
$$= 4 \int_0^2 (8 - 2x^2) y - \tfrac{2}{3} y^3 \Big|_0^{\sqrt{4-x^2}} dx$$
$$= 4 \int_0^2 \left[(8 - 2x^2) - \tfrac{2}{3}(4 - x^2) \right] \cdot \sqrt{4 - x^2}\, dx.$$
$$= 4 \int_0^2 \left[(4 - x^2) \cdot \tfrac{4}{3} \right] \cdot \sqrt{4 - x^2}\, dx.$$
$$= \tfrac{16}{3} \int_0^2 (4 - x^2)^{\frac{3}{2}}\, dx$$
$$= \tfrac{16}{3} \int_0^{\frac{\pi}{2}} 2 \cos^4\theta \cdot 2 \cos\theta\, d\theta$$
$$= \tfrac{32}{3} \int_0^{\frac{\pi}{2}} \cos^5\theta\, d\theta = 16\pi.$$

Double integral in Polar coordinate.

$\iint_S f(x,y)\, dA$. $S = \{(r,\theta) \mid a \le r \le b,\ c \le \theta \le d\}$.

$\Delta r = \frac{b-a}{m}$. $\Delta\theta = \frac{d-c}{n}$



$$\Delta A_i = \tfrac{1}{2}(r_i^2 + \tfrac{1}{2}\Delta r)^2 \cdot \Delta\theta - \tfrac{1}{2}(r_i^2 - \tfrac{1}{2}\Delta r)^2 \cdot \Delta\theta.$$
$$= \tfrac{1}{2} \cdot 2 \cdot r_i^2 \Delta r \Delta\theta = r_i^2 \Delta r \Delta\theta$$

$$\therefore V = \iint_S f(x,y)\, dA = \lim_{n\to\infty} \sum_{i=1}^{\infty} f(r_i^* \cdot \theta_i^*)\, r_i^* \Delta r \Delta\theta$$

$$\therefore V = \int_c^d \left(\int_a^b f(r,\theta)\, r\, dr \right) d\theta$$

e.x. find the volume of a sphere with radius α

Sphere: $x^2 + y^2 + z^2 = \alpha^2$

$z = \sqrt{\alpha^2 - (x^2 + y^2)} = f(x,y)$.

$V = 2 \iint_S f(x,y)\, dA$.

Let $x = r\cos\theta$, $y = r\sin\theta$

$f(r,\theta) = \sqrt{\alpha^2 - r^2}$.

$$V = 2 \int_0^{2\pi} \int_0^{\alpha} f(r,\theta) \cdot r\, dr\, d\theta.$$
$$= 2 \int_0^{2\pi} \int_0^{\alpha} \sqrt{\alpha^2 - r^2} \cdot r\, dr\, d\theta.$$
$$= \int_0^{2\pi} \int_0^{\alpha} \sqrt{\alpha^2 - r^2}\, dr^2\, d\theta.$$
$$= \int_0^{2\pi} -\tfrac{2}{3}(\alpha^2 - r^2)^{\frac{3}{2}} \Big|_0^{\alpha}\, d\theta.$$
$$= \int_0^{2\pi} \tfrac{2}{3}[\alpha^2]^{\frac{3}{2}}\, d\theta$$
$$= \tfrac{2}{3}\alpha^3 \cdot 2\pi = \tfrac{4}{3}\pi\alpha^3$$

Exercise: Find the volume beneath the graph of $z = f(x,y) = xy + 10$ and above the annular region $S = \{(r,\theta) \mid 2 \le r \le 4,\ 0 \le \theta \le 2\pi\}$.

Let: $x = r\cos\theta$, $y = r\sin\theta$

$z = g(r,\theta) = r^2 \sin\theta \cos\theta + 10$.

$$V = \int_0^{2\pi} \int_2^4 (r^2 \sin\theta + 10)\, r\, dr\, d\theta.$$
$$= \int_0^{2\pi} \tfrac{1}{2} \int_2^4 r^3 \sin\theta + 10 r\, dr^2\, d\theta.$$
$$= \int_0^{2\pi} \left[\tfrac{1}{2} \cdot \tfrac{1}{4} r^4 \sin\theta + 5r^2 \right]_2^4 d\theta$$
$$= \int_0^{2\pi} \left[\tfrac{1}{4} r^4 \sin\theta + 5r^2 \right]_2^4 d\theta.$$
$$= \int_0^{2\pi} 60 \sin\theta + 60\, d\theta = -60\cos\theta + 60\theta \Big|_0^{2\pi}$$
$$= 120\pi$$

Double integral over general polar regions

$\Omega = \{(r,\theta) \mid 0 \le \theta \le d,\ 0 \le g(\theta) \le r \le h(\theta)\}$.

$$\iint_\Omega f(r,\theta)\, dA = \int_c^d \left(\int_{g(\theta)}^{h(\theta)} f(r,\theta)\, r\, dr \right) d\theta.$$

Example: Let $f(x,y) = \frac{y}{\sqrt{x^2 + y^2}}$ on Ω, the region in the first quadrant bounded by the circle $r = 2$ and the cardioid $r = 2(1 + \cos\theta)$. Find the value of $\iint_\Omega f(x,y)\, dA$.

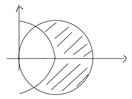
Let $x = r\cos\theta$, $y = r\sin\theta$.

$g(r,\theta) = \frac{r\sin\theta}{r} = \sin\theta$.

$$\iint_\Omega f(x,y)\, dA = \int_0^{\frac{\pi}{2}} \int_2^{2(1+\cos\theta)} \sin\theta\, r\, dr\, d\theta.$$
$$= \int_0^{\frac{\pi}{2}} \tfrac{1}{2} \sin\theta\, r^2 \Big|_2^{2(1+\cos\theta)} d\theta.$$
$$= \int_0^{\frac{\pi}{2}} \tfrac{1}{2} \sin\theta \cdot [4 + 4\cos\theta + 4\cos^2\theta]\, d\theta.$$
$$= -\int_0^{\frac{\pi}{2}} 4\cos\theta + 2\cos^2\theta\, d\cos\theta$$
$$= -2\cos^2\theta - \tfrac{2}{3}\cos^3\theta \Big|_0^{\frac{\pi}{2}}$$
$$= 2 + \tfrac{2}{3} = \tfrac{8}{3}.$$

Exercise: Let Ω be a region such that it is outside $r = 2$ and inside $r = 4\cos\theta$. Suppose $f(r,\theta) \ge 0$ on Ω. Express the volume of the solid under the graph of f over Ω as an iterated integral.

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_2^{4\cos\theta} f(r,\theta)\, r\, dr\, d\theta$$



Gaussian Integral

"$\int_{-\infty}^{\infty} e^{-x^2}\, dx$"

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = 2 \cdot \int_0^{\infty} e^{-x^2}\, dx.$$

Let

$f(r,\theta) = e^{-r^2}$, $R = \{(r,\theta) \mid 0 \le r < \infty,\ 0 \le \theta \le \frac{\pi}{2}\}$.

$$\int_0^{\infty} e^{-x^2}\, dx = \iint_R f(r,\theta)\, dA = \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta.$$
$$= \int_0^{\frac{\pi}{2}} -\tfrac{1}{2} e^{-r^2} \Big|_0^{\infty} d\theta$$
$$= \int_0^{\frac{\pi}{2}} \tfrac{1}{2}\, d\theta = \tfrac{\pi}{4}$$

Let $f(x,y) = e^{-(x^2+y^2)}$ where $x = r\sin\theta$, $y = r\cos\theta$.

$\Rightarrow f(x,y) = f(r,\theta)$

$\iint_R f(x,y)\, dA$, $R = \{(x,y) \mid x \ge 0,\ y \ge 0\}$.

$$= \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy.$$
$$= \int_0^{\infty} \int_0^{\infty} e^{-x^2} \cdot e^{-y^2}\, dx\, dy.$$
$$= \int_0^{\infty} e^{-y^2} \int_0^{\infty} e^{-x^2}\, dx\, dy.$$
$$= \int_0^{\infty} e^{-y^2}\, dy \cdot \int_0^{\infty} e^{-x^2}\, dx$$
$$= \left[\int_0^{\infty} e^{-x^2}\, dx \right]^2 = \iint_R f(r,\theta) = \tfrac{\pi}{4}.$$
$$\therefore \int_0^{\infty} e^{-x^2}\, dx = \tfrac{\sqrt{\pi}}{2}$$
$$\therefore \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Triple Integral

Consider $w = f(x,y,z)$ defined on a closed and bounded domain D in $\mathbb{R}^3$. Then D can be contained in a cuboid $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] = \{(x,y,z) \mid a_1 \le x \le b_1,\ a_2 \le y \le b_2,\ a_3 \le z \le b_3\}$. We divide the cuboid into small sub-cuboids by planes parallel to xy, yz, and xz planes. Choose the small sub-cuboids that are contained in D and label them from $k = 1$ to $k = N$. For the kth sub-cuboid, we denote its side lengths to be Δx_k, Δy_k, and Δz_k. Then its volume ΔV_k is $\Delta x_k \Delta y_k \Delta z_k$. Moreover, we select an arbitrary point (x_k^*, y_k^*, z_k^*) in the sub-cuboid.

The **triple integral** of f over D is defined to be the limit of the following "Riemann sum":

$$\iiint_D f(x,y,z)\, dV = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k^*, y_k^*, z_k^*) \Delta V_k$$

Suppose D is a 3D solid and its density varies at different points of the solid according to $f(x,y,z)$. Then $\iiint_D f(x,y,z)\, dV$ = Total mass of D.

In particular, if $f(x,y,z) = 1$, i.e. the solid has "uniform density" of 1, we can regard the triple integral $\iiint_D 1\, dV$ as the volume of the solid.

e.x. Find the value of $\iiint_{G_1} xy^2 z^3\, dV$ where $G_1 = [0,2] \times [-1,0] \times [0,3]$.

$$\iiint_{G_1} xy^2 z^3\, dV = \int_0^2 \int_{-1}^0 \int_0^3 xy^2 z^3\, dz\, dy\, dx.$$
$$= \int_0^2 \int_{-1}^0 \tfrac{1}{2} x^2 y^2 z^3 \Big|_0^3 dy\, dz.$$
$$= \int_0^2 \int_{-1}^0 \tfrac{9}{2} x^2 y^2 z^3\, dy\, dz.$$
$$= \int_0^2 -\tfrac{3}{2} y^3 z^3 \Big|_{-1}^0 dz.$$
$$= \int_0^2 \tfrac{3}{2} z^3\, dz.$$
$$= \tfrac{3}{12} \cdot z^4 \Big|_0^2 = \tfrac{3}{12} \cdot 16 = \tfrac{8}{3}.$$

e.x. Find the mass of cuboid $D = [0,3] \times [0,2] \times [0,1]$ with density $f(x,y,z) = 2x + 3yz$.

$$M = \int_0^3 \int_0^2 \int_0^1 2x + 3yz\, dz\, dy\, dx.$$
$$= \int_0^3 \int_0^2 [2xz + \tfrac{3}{2} y z^2]_0^1\, dy\, dx.$$
$$= \int_0^3 \int_0^2 2x + \tfrac{3}{2} y\, dy\, dx.$$
$$= \int_0^3 [2xy + \tfrac{3}{4} y^2]_0^2\, dx$$
$$= \int_0^3 4x + 3\, dx$$
$$= 2x^2 + 3x \Big|_0^3 = 18 + 9 = 27.$$

Triple integral over a general domain

Suppose $w = f(x,y,z)$ is defined on a domain D, which is solid bounded by the graph of $z = G(x,y)$ from below and by the graph of $z = H(x,y)$ from above over a region R on the xy plane. In other words, $H(x,y) \ge G(x,y)$ for all (x,y) in R. This kind of domains is called a **simple xy-solid**. We can reduce the triple integral over a simple xy-solid to the double integral as follows:

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{G(x,y)}^{H(x,y)} f(x,y,z)\, dz \right) dA$$

Moreover, if R is a Type I region bounded from below by the graph of $y = g(x)$ and from above by the graph of $y = h(x)$ over $[a,b]$, we can express the triple integral as the following iterated integral:

$$\iiint_D f(x,y,z)\, dV = \int_a^b \left(\int_{g(x)}^{h(x)} \left(\int_{G(x,y)}^{H(x,y)} f(x,y,z)\, dz \right) dy \right) dx$$

We have similar results for **simple xz-solid** and **simple yz-solid**.

Example: Let G be the wedge in the first octant cut the set of points in $\mathbb{R}^3$ such that x,y,z coordinates are non-negative, that is cut from the cylindrical solid $y^2 + z^2 \le 1$ by the planes $y = x$ and $x = 0$. Find the value of the triple integral $\iiint_G z\, dV$.

$z = \sqrt{1 - y^2}$

$$\iiint_G z\, dV = \iint_R \left(\int_0^{\sqrt{1-y^2}} z\, dz \right) dA, \text{ where } R \text{ is type II region.}$$
$$= \int_0^1 \int_0^y \left[\int_0^{\sqrt{1-y^2}} z\, dz \right] dx\, dy$$
$$= \int_0^1 \int_0^y \left[\tfrac{1}{2} z^2 \right]_0^{\sqrt{1-y^2}} dx\, dy.$$
$$= \int_0^1 \int_0^y \tfrac{1}{2}(1 - y^2)\, dx\, dy.$$
$$= \int_0^1 \tfrac{1}{2}(1 - y^2) \cdot y\, dy.$$
$$= -\tfrac{1}{8}(1 - y^2)^2 \Big|_0^1$$
$$= \tfrac{1}{8}.$$

Example: Find the volume of the solid G bounded by the two paraboloids $y = x^2 + z^2$ and $y = 16 - 3x^2 - z^2$.

$x^2 + z^2 = 16 - 3x^2 - z^2$.

$4x^2 + 2z^2 - 16 = 0$

$2x^2 + z^2 - 8 = 0$. $z = \sqrt{8 - 2x^2}$

$$\iint_R \int_{x^2+z^2}^{16-3x^2-z^2} 1\, dy.$$
$$= \int_{-2}^{2} \int_{-\sqrt{8-2x^2}}^{\sqrt{8-2x^2}} \int_{x^2+z^2}^{16-3x^2-z^2} 1\, dy\, dz\, dx.$$
$$= \int_{-2}^{2} \int_{-\sqrt{8-2x^2}}^{\sqrt{8-2x^2}} (16 - 4x^2 - 2z^2)\, dz\, dx.$$
$$= \int_{-2}^{2} \left[16z - 4x^2 z - \tfrac{2}{3} z^3 \right]_{-\sqrt{8-2x^2}}^{\sqrt{8-2x^2}} dx.$$
$$= \int_{-2}^{2} 32\sqrt{8-2x^2} - 8x^2\sqrt{8-2x^2} - \tfrac{16}{3}(8 - 2x^2)^{\frac{3}{2}}\, dx.$$
$$= \int_{-2}^{2} \left[32 - 8x^2 - \tfrac{16}{3}(8 - 2x^2) \right] \sqrt{8 - 2x^2}\, dx.$$
$$= \int_{-2}^{2} \left[32 - 8x^2 - \tfrac{32}{3} + \tfrac{8}{3}x^2 \right] \sqrt{2}\sqrt{4 - x^2}\, dx.$$
$$= \tfrac{\sqrt{2}\, 32}{3} \int_{-2}^{2} [4 - x^2]^{\frac{3}{2}}\, dx.$$
$$= \tfrac{32\sqrt{2}}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta = 8\sqrt{2}\pi$$

Changing integration order

$$\int_0^1 \left(\int_x^1 \int_0^y e^{x^3}\, dy \right) dx \Big) dz.$$
$$= \int_0^1 \int_0^x \int_0^y e^{x^3}\, dy\, dz\, dx$$
$$= \int_0^1 x^3 e^{x^3}\, dx$$
$$= \tfrac{1}{3}(e - 1)$$

Triple integrals in cylindrical coordinates

Suppose we want to compute the triple integral $\iiint_G f(x,y,z)\, dV$, where G is a simple xy-solid over a general polar region $R = \{(r,\theta) \mid c \le \theta \le d,\ g(\theta) \le r \le h(\theta)\}$. It is more convenient to change the triple integral into cylindrical coordinates as follows:

$$\iiint_G f(x,y,z)\, dV = \int_c^d \left(\int_{g(\theta)}^{h(\theta)} \left(\int_{G(r,\theta)}^{H(r,\theta)} f(r,\theta,z)\, dz \right) r\, dr \right) d\theta$$

e.x. Find volume of cylinder $x^2 + y^2 = 9$ between planes $x + z = 5$ and $z = 1$

$x = r\cos\theta$. $z = H(r,\theta) = 5 - r\cos\theta$.

$z = G(r,\theta) = 1$

$$V = \iiint_{G_1} f(x,y,z)\, dV = \int_0^{2\pi} \left(\int_0^3 \left(\int_1^{5 - r\cos\theta} 1\, dz \right) r\, dr \right) d\theta$$
$$= \int_0^{2\pi} \int_0^3 [z]_1^{5 - r\cos\theta} \cdot r\, dr\, d\theta.$$
$$= \int_0^{2\pi} \int_0^3 4r - r^2\cos\theta\, dr\, d\theta.$$
$$= \int_0^{2\pi} \left[2r^2 - \tfrac{1}{3} r^3 \cos\theta \right]_0^3 d\theta.$$
$$= \int_0^{2\pi} [18 - 9\cos\theta]\, d\theta$$
$$= 36\pi.$$

Triple integrals in spherical coordinates

Sometimes we might want to express a triple integral in spherical coordinates. Suppose $f(x,y,z)$ is defined on a "spherical box" $B = \{(\rho,\theta,\phi) \mid a \le \rho \le b,\ c \le \theta \le d,\ p \le \phi \le q\}$. We then divide each interval into l, m, n subdivisions such that $\Delta\rho = \frac{b-a}{l}$, $\Delta\theta = \frac{d-c}{m}$, $\Delta\phi = \frac{q-p}{n}$. Label the spherical subbox from $k = 1$ to $k = N$. Let $(\rho_k^*, \theta_k^*, \phi_k^*)$ be the centre point in the kth spherical subbox. The volume ΔV_k of the kth spherical subbox equals $(\Delta\rho_k)(\rho_k^*\Delta\theta_k)(\rho_k^* \sin(\phi_k^*)\, \Delta\phi) = (\rho_k^*)^2 \sin(\phi_k^*)\, \Delta\rho\Delta\theta\Delta\phi$, as shown in the diagram below.

The triple integral in spherical coordinates is the limit of the following Riemann sum:

$$\iiint_E f(x,y,z)\, dA = \lim_{N\to\infty} \sum_{k=1}^{N} f(\rho_k^*, \theta_k^*, \phi_k^*)(\rho_k^*)^2 \sin(\phi_k^*)\, \Delta\rho\Delta\theta\Delta\phi$$

Therefore, we can rewrite the triple integral as an iterated integral in spherical coordinates as follows:

$$\iiint_E f(x,y,z)\, dV = \int_a^b \left(\int_c^d \left(\int_p^q f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\phi \right) d\theta \right) d\phi$$

Example: Find the volume of the solid T bounded from above and below by the sphere $x^2 + y^2 + z^2 = 16$ and the cone $z = \sqrt{x^2 + y^2}$.

$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \left(\int_0^4 \rho^2 \sin\phi\, d\rho \right) d\theta\, d\phi$$
$$= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \left(\tfrac{1}{3}\rho^3 \sin\phi \right)_0^4 d\theta\, d\phi$$
$$= \int_0^{\frac{\pi}{4}} \left[\tfrac{64}{3} \sin\phi \cdot \theta \right]_0^{2\pi} d\phi$$
$$= \tfrac{128}{3}\pi \cdot \cos\theta \Big|_0^{\frac{\pi}{4}}$$
$$= \tfrac{128}{3}\pi \left(-\tfrac{\sqrt{2}}{2} + 1 \right)$$
$$= \tfrac{64}{3}\pi(2 - \sqrt{2})$$

1. **Rectangular (Cartesian) coordinates**: A point is specified by the coordinates (x,y,z), where x, y and z are real numbers indicating the projection of the point onto three axes.

2. **Cylindrical coordinates**: A point is specified by the coordinates (r,θ,z), where z is the z-coordinate in rectangular coordinates and (r,θ) are the polar coordinates of the projection of the point onto xy plane. Note that here we assume $r \ge 0$ and $0 \le \theta \le 2\pi$.

3. **Spherical coordinates**: A point is specified by the coordinates (ρ,θ,ϕ), where θ is the θ-coordinate in cylindrical coordinates, ρ is the distance between the point and the origin, and ϕ is the angle measured from z-axis to the line segment joining the point to the origin. Note that $\rho \ge 0$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$

R ← Rectangular. C ← Cylindrical. S ← Spherical

$C \to R$: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$R \to C$. $\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(\frac{y}{x}) \\ z = z \end{cases}$

$S \to R$ $\begin{cases} x = \rho \cdot \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta. \\ z = \rho\cos\theta. \end{cases}$

$R \to S$ $\begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\frac{y}{x}) \\ \varphi = \cos^{-1}(\frac{z}{\sqrt{x^2+y^2+z^2}}) \end{cases}$

$C \to S$ $\begin{cases} \rho = \sqrt{r^2 + z^2} \\ \theta = \theta \\ \phi = \tan^{-1}(\frac{r}{z}) \end{cases}$

$S \to C$: $\begin{cases} r = \rho\sin\phi. \\ \theta = \theta. \\ z = \rho\cos\phi. \end{cases}$

e.x. $P = (1, -2, 3)$, find cylindrical and spherical Coordinate of P.

$P_c = (\sqrt{5}, \tan^{-1}(-2), 3)$

$P_s = (\sqrt{14}, \tan^{-1}(-2), \cos^{-1}(\frac{3}{\sqrt{14}}))$