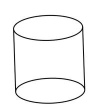
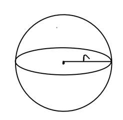
Surface/Graphs/Level Curves/Level Surfaces

Cylinders:



$x^2+y^2=r^2$ (x, y are independent from z)

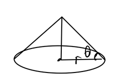
Spheres



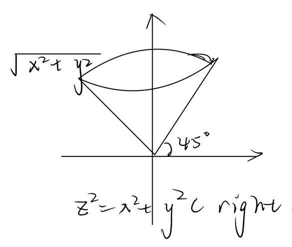
$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$

if compress/stretch ⇒ "ellipsoid"

Cones:



$z=\sqrt{x^2+y^2}$



$z^2=x^2+y^2$ (right cone)

$a^2z^2=x^2+y^2$ Cones

Multivariables functions

$z=f(x,y)$ (domain $D: \mathbb{R}^2$)

level curves $f(x,y)=k$

level surface $f(x,y,z)=k$

Limits and Continuity

$$L=\lim_{x,y\to x_0,y_0} f(x,y)$$

Actual (Rigorous) Definition:

For any small positive real number ε, we can find another small positive real number $\delta>$

Such that: whenever distance from (x,y) to (x_0,y_0), $|f(x,y)-L|<\varepsilon$.

Limit along a curve

" $\lim_{(x,y)\to(x_0,y_0)} f(x,y)=\,?$ " ⇒ There are many different ways approach (x_0,y_0)!

Suppose a curve C: $\vec{r}(t)=\langle x(t),y(t)\rangle$ ⇒ $\vec{r}(t_0)=\langle x_0,y_0\rangle$

$$\boxed{\text{Definite: } \lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{along } C}} f(x,y)=\lim_{t\to t_0} f(x(t),y(t))}$$

Existence

for regular function: $\lim_{x\to x_0} f(x)=L$ iff $\lim_{x\to x_0^-} f(x)=L$ & $\lim_{x\to x_0^+} f(x)=L$

Similarly:

for multivariable function (let's say 2-variable)

$$\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L \text{ iff } \lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{along } C}} f(x,y)=L \text{ for any curve } C}$$

Property of limits of multivariable function is similar with single variable function

- Addition
- scalar multiplication
- multiplication / division
- product
- ...

Continuity

$f(x,y)$ is continuous at (a,b) if:

$$\lim_{(x,y)\to(a,b)} f(x,y)=(a,b)$$

Limit along a curve

C: $\vec{r}(t)=\langle x(t),y(t)\rangle$

$\vec{r}(t_0)=\langle x(t_0),y(t_0)\rangle=\langle x_0,y_0\rangle$

$\Rightarrow \lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{along } C}} f(x,y)=\lim_{t\to t_0} f(x(t),y(t))$

$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$ iff $\lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{along } C}} f(x,y)=L$ for any C

ex. $f(x,y)=\frac{3xy^2}{x^2+y^4}$. Show that $\lim_{(x,y)\to(0,0)}$ does not exist.

C_1: $r(t)=\langle 0,t\rangle$ $r(0)=(0,0)$

$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_1}}=\lim_{t\to0}\frac{0}{t^4}=0$.

C_2: $r(t)=\langle t^2,t\rangle$ $r(0)=(0,0)$

$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_2}}=\lim_{t\to0}\frac{3t^4}{2t^4}=\frac{3}{2}$.

$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_1}} \neq \lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_2}}$ hence limit doesn't exist

Property of limit ⇒ Similar with regular $\lim f(x)$

ex. $\lim_{(x,y)\to(2,8)} (2x^3y-(xy)^{\frac{1}{4}})$

$=\lim_{(x,y)\to(2,8)} 2x^3y-\lim_{(x,y)\to(2,8)}(xy)^{\frac{1}{4}}$

$=\lim_{(x,y)\to(2,8)} 2\cdot64-\lim_{(x,y)\to(2,8)}(16)^{\frac{1}{4}}$

$=128-2=126$

ex. $\lim_{(x,y)\to(4,1)}\frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}}$

$=\lim_{(x,y)\to(4,1)}\frac{(xy-4y^2)(\sqrt{x}+2\sqrt{y})}{(\sqrt{x}-2\sqrt{y})(\sqrt{x}+2\sqrt{y})}$

$=\lim_{(x,y)\to(4,1)}\frac{y(x-4y)(\sqrt{x}+2\sqrt{y})}{x-4y}$

$=\lim_{(x,y)\to(4,1)} y(\sqrt{x}+2\sqrt{y})=4$

ex. $\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^2+y^2}$ we can use polar coordinate

Let $x=r\cos\theta$, $y=r\sin\theta$, where $r=\sqrt{x^2+y^2}$

$\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^2+y^2}=\lim_{r\to0^+}\frac{r^2\cos^2\theta\cdot r\sin\theta}{r^2}=\lim_{r\to0^+} r\sin\theta\cos^2\theta$.

Since: $-r\le r\sin\theta\cos^2\theta\le r$ "Squeeze theorem"

$\lim_{r\to0^+}(-r)\le\lim_{r\to0^+} r\sin\theta\cos^2\theta\le\lim_{r\to0^+} r$

$\Rightarrow \lim_{r\to0^+} r\sin\theta\cos^2\theta=0$, Hence $\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^2+y^2}=0$

e.x. $\lim_{(x,y)\to(0,0)}\frac{(x^2+y^2)^2}{\cos(x^2+y^2)-1}$

let $x=r\cos\theta$, $y=r\sin\theta$ where $r=\sqrt{x^2+y^2}$

$\lim_{(x,y)\to(0,0)}\frac{(x^2+y^2)^2}{\cos(x^2+y^2)-1}=\lim_{r\to0^+}\frac{r^4}{\cos(r^2)-1}$

$=\lim_{r\to0^+}\frac{4r^2}{-\sin(r^2)\cdot2}=\lim_{r\to0^+}\frac{8r}{-4r\cos(r^2)}=-2$.

e.x. $\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}=\lim_{r\to0^+}\frac{r^2\cos\theta\sin\theta}{r^2}=\lim_{r\to0^+}\cos\theta\sin\theta$

∴ limit does not exist

Partial Derivative

$$f(x,y),\quad \frac{\partial f(x,y)}{\partial x}=\lim_{h\to0}\frac{f(x+h,y)-f(x,y)}{h}$$

All the differentiation rules for ordinary derivatives can be used to compute partial derivatives:

To compute f_x, y is treated as a constant and differentiate with respect to x.

To compute f_y, x is treated as a constant and differentiate with respect to y.

e.x $f(x,y)=4x^2y^3-3x\sin y+e^{xy}$

$$\begin{cases} f_x=8xy^3-3\sin y+ye^{xy}\\ f_y=12x^2y^2-3x\cos y+xe^{xy}\end{cases}$$

e.x. $g(x,y,z)=x^3y^2z-6xz\cos(zy)$

$$\begin{cases} g_x=3x^2y^2z-6z\cos(zy)\\ g_y=2x^3yz-6xz(-\sin(zy)\cdot z)\\ \quad=2x^3yz+6xz^2\sin(zy)\\ g_z=x^3y^2-6x\cdot(\cos(zy)-zy\sin(zy))\\ \quad=x^3y^2-6x\cos(zy)+6xyz\sin(zy)\end{cases}$$

e.x. $g(x,y,z)=e^{xyz}\sin(xy)-y\ln(2x^2+z^4)$

$g_x=yze^{xyz}\sin(xy)+e^{xyz}\cdot y\cdot\cos(xy)-y\frac{1}{2x^2+z^4}\cdot4x$

$g_y=xze^{xyz}\sin(xy)+e^{xyz}x\cos(xy)-\ln(2x^2+z^4)$

$g_z=xye^{xyz}\sin(xy)-y\cdot\frac{4z^3}{2x^2+z^4}$

e.x. $z=f(x,y)=3x^2y+4y^3$, find $f_x(1,2)$

$f_x=6xy$, $f_x(1,2)=12$

Implicit partial differentiation

for $x^2+y^2+z^2=1$ slope in x-direction ⇔ $\frac{\partial z}{\partial x}$

$2x+2z\cdot\frac{\partial z}{\partial x}=0$ ⇒ $\frac{\partial z}{\partial x}=-\frac{x}{z}$

Differentiability

$f'(x)=\lim_{h\to0}\frac{f(x_0+h)-f(x_0)}{h}$ ⇔ $\lim_{\Delta x\to0}\frac{\overbrace{f(x_0+\Delta x)-f(x_0)-f'(x)\Delta x}^{E(\Delta x)}}{\Delta x}=0$

$\lim_{\Delta x\to0}\frac{E(\Delta x)}{\Delta x}=0$

$E(\Delta x,\Delta y)=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)-(f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y)$

$f(x,y)$, f is differentiable at (x_0,y_0) if $\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{E(\Delta x,\Delta y)}{|\langle\Delta x,\Delta y\rangle|}=0$

e.x. $f(x,y)=|xy|$, prove it is differentiable at $(0,0)$ but not diff at $(1,0)$

$$\begin{cases} f_x(0,0)=\lim_{\Delta x\to0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=0\\ f_y(0,0)=0\end{cases}$$

$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{E(\Delta x,\Delta y)}{|\langle\Delta x,\Delta y\rangle|}=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)-(f_x\cdot\Delta x+f_y\Delta y)}{\sqrt{\Delta x^2+\Delta y^2}}$

$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{|\Delta x\cdot\Delta y|-0-0}{\sqrt{\Delta x^2+\Delta y^2}}$

$=\lim_{r\to0^+}\frac{r^2\cos\theta\sin\theta}{r}=0$ ("Squeeze theorem")

Theorem: Let $f(x,y)$ be a function such that all its partial derivatives f_x and f_y exist and continuous at (x_0,y_0), then f is differentiable at (x_0,y_0).

Theorem: If f is differentiable at (x_0,y_0), then f is continuous at (x_0,y_0).

Chain Rule:

e.x. Given that $z=2\cos x\sin(xy)$ $x=e^t$ $y=\ln t$.

Find $\frac{dz}{dt}$.

$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$.

$\frac{dx}{dt}=e^t$ $\frac{dy}{dt}=\frac{1}{t}$.

$\frac{dz}{dt}=2\cdot(-\sin x\sin(xy)+\cos x\,y\cos(xy))\cdot e^t+2\cos x\cdot\cos(xy)\cdot x\cdot\frac{1}{t}$.

$=-2\sin e^t\sin(e^t\ln t)+2\cos e^t\cdot e^t\ln t\cdot\cos(e^t\ln t)+2\cos e^t\cos e^t\ln t\cdot e^t\cdot\frac{1}{t}$.

e.x. $z=\sin(2x)\cos(3y)$ $x=\frac{s+t}{2}$ $y=\frac{s-t}{2}$, Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

$$\begin{cases}\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}\\ \frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\end{cases}$$

where $\frac{\partial x}{\partial s}=\frac{\partial x}{\partial t}=\frac{1}{2}$, $\frac{\partial y}{\partial s}=\frac{1}{2}$ $\frac{\partial y}{\partial t}=-\frac{1}{2}$

$\frac{\partial z}{\partial s}=2\sin(2x)\cdot\cos(3y)\cdot\frac{1}{2}-\sin(2x)\cdot3\sin(3y)\cdot\frac{1}{2}$

$=2\sin(s+t)\cdot\cos(\frac{3}{2}(s-t))\cdot\frac{1}{2}-\sin(s+t)\cdot3\sin(\frac{3}{2}(s-t))\cdot\frac{1}{2}$

$\frac{\partial z}{\partial s}=2\sin(2x)\cdot\cos(3y)\cdot\frac{1}{2}-\sin(2x)\cdot3\sin(3y)\cdot(-\frac{1}{2})$

$=2\sin(s+t)\cdot\cos(\frac{3}{2}(s-t))\cdot\frac{1}{2}-\sin(s+t)\cdot3\sin(\frac{3}{2}(s-t))\cdot(-\frac{1}{2})$

Directional Derivative

$\vec{u}=\langle u_1,u_2\rangle$ be an unit vector.

$D_{\vec{u}}f(x_0,y_0)=\lim_{t\to0}\frac{f(x_0+u_1t,y_0+u_2t)-f(x_0,y_0)}{t}$

$=\frac{d}{dt}f(x_0+tu_1,y_0+tu_2)\Big|_{t=0}$

Theorem: $D_{\vec{u}}f(x_0,y_0)=\langle f_x(x_0,y_0),f_y(x_0,y_0)\rangle\cdot\vec{u}$.

proof: Chain Rule: $\frac{d}{dt}f(x_0+tu_1,y_0+tu_2)$

$=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$

$=f_x(x_0,y_0)\cdot u_1+f_y(x_0,y_0)\cdot u_2$

e.x. $w=g(x,y,z)=\frac{3}{4}x^2+\frac{1}{2}y^2+3z^2$ $\vec{v}=\langle\frac{1}{3},-\frac{2}{3},\frac{2}{3}\rangle$

Find $D_{\vec{v}}g(1,2,3)$

☆ first. We need to check if $\vec{v}$ is unit vector.

if not. $\vec{v}'=\frac{\vec{v}}{|\vec{v}|}$

$$\begin{cases} g_x=\frac{3}{2}x, & g_x(1,2,3)=\frac{3}{2}\\ g_y=y, & g_x(1,2,3)=2\\ g_z=6z, & g_x(1,2,3)=18\end{cases}$$

⇒ $D_{\vec{v}}g(1,2,3)=\frac{1}{3}\cdot\frac{3}{2}+2\cdot(-\frac{2}{3})+18\cdot\frac{2}{3}$

$=\frac{1}{2}-\frac{4}{3}+12=\frac{67}{6}$

Remark: For any unit vector $\vec{u}$ in $\mathbb{R}^2$, it can be written as $\vec{u}=\langle\cos\phi,\sin\phi\rangle$, where ϕ is the angle measured from the positive x-axis to the direction of $\vec{u}$ in anticlockwise direction. Therefore, we have the following formula for directional derivatives in terms of ϕ:

$$D_{\vec{u}}f(x_0,y_0)=f_x(x_0,y_0)\cos\phi+f_y(x_0,y_0)\sin\phi$$

Gradient $\nabla f(x,y)=\langle f_x(x,y),f_y(x,y)\rangle$

⇒ $D_{\vec{u}}f(x,y)=\nabla f(x,y)\cdot\vec{u}$

if $\nabla f(x,y)=\langle0,0\rangle$ then all directional derivative $=0$.

Geometric Meaning

$D_{\vec{u}}f(x,y)=|\nabla f(x,y)|\cdot\cos\theta$. where θ is angle between $\vec{u}$ and Gradient

⇒ When $\theta=0$. $D_{\vec{u}}f(x,y)$ has maximum $|\nabla f(x,y)|$

(i.e. When $\vec{u}$ & $\nabla f(x,y)$ are in Same Direction)

$D_{\vec{u}}f(x,y)$ has minimum $-|\nabla f(x,y)|$ when $\vec{u}$ & $\nabla f(x,y)$ are in opposite direction

e.x. $f(x,y)=5+x^2+3y^2$. Find the direct of the steepest ascent and descent on $z=f(x,y)$ when $(x,y)=(1,2)$

Find maximum rate of the steepest ascent.

$\nabla f(x,y)=\langle f_x,f_y\rangle$

$=\langle2x,6y\rangle$

$\nabla f(1,2)=\langle2,12\rangle$ ∴ ascent: $\langle2,12\rangle$ decent: $\langle-2,-12\rangle$

rate $=|\nabla f(1,2)|=2\sqrt{37}$

e.x. $z=f(x,y)=\sqrt{1+2x^2+y^2}$. find equation of line tangent to the level curve at $(1,1)$.

$\nabla f(x,y)=\langle f_x,f_y\rangle$.

$f_x=\frac{2x}{\sqrt{1+2x^2+y^2}}$ $f_y=\frac{y}{\sqrt{1+2x^2+y^2}}$

$\nabla f(1,1)=\langle 1,\frac{1}{2}\rangle$

line: $\langle1,\frac{1}{2}\rangle\cdot\langle x-1,y-1\rangle=0$. ⇒ $2x+y-3=0$

Suppose $z=f(x,y)$ is a differentiable function and k is any real number. Then the equation of the level curve of $z=f(x,y)$ when $z=k$ is $f(x,y)=k$. Suppose such level curve is parametrized by $\vec{r}(t)=\langle x(t),y(t)\rangle$. Then we have

$$f(x(t),y(t))=k$$

Differentiate both sides and use chain rule, we have

$$f_x(x(t),y(t))x'(t)+f_y(x(t),y(t))y'(t)=0$$
$$\Longrightarrow \nabla f(x(t),y(t))\cdot\vec{r}'(t)=0$$

Therefore, $\nabla f(x(t),y(t))$ is orthogonal to the tangent vector $\vec{r}'(t)$ of the level curve for all t.

$\nabla g(x(t),y(t),z(t))$ is orthogonal to tangent vector at $(x(t),y(t),z(t))$

⇒ ∇g is normal vector of level surface

e.x. Find the equation of tangent plane of a unit sphere centered at origin at point $(\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3})$

$g(x,y,z)=x^2+y^2+z^2$

$\nabla g(x,y,z)=\langle2x,2y,2z\rangle$.

$g(\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3})=\langle\frac{2}{\sqrt3},\frac{2}{\sqrt3},\frac{2}{\sqrt3}\rangle$, tangent plane:

$\frac{2}{\sqrt3}(x-\frac{1}{\sqrt3})+\frac{2}{\sqrt3}(y-\frac{1}{\sqrt3})+\frac{2}{\sqrt3}(z-\frac{1}{\sqrt3})=0$

We can use the following trick: Define the function of three variables $g(x,y,z)=z-f(x,y)$. Then the graph of $z=f(x,y)$ can be regarded as the level surface $g(x,y,z)=0$. Let $z_0=f(x_0,y_0)$. In the previous section, we know that the equation of the tangent plane to the level surface $g(x,y,z)=0$ at (x_0,y_0,z_0) is as follows:

$$g_x(x_0,y_0,z_0)(x-x_0)+g_y(x_0,y_0,z_0)(y-y_0)+g_z(x_0,y_0,z_0)(z-z_0)=0$$

By computation, we have

$$g_x=-f_x,\ g_y=-f_y,\ g_z=1$$

i.e. $\nabla g(x_0,y_0,z_0)=\langle-f_x(x_0,y_0),-f_y(x_0,y_0),1\rangle$ (normal vector to the tangent plane)

Therefore, we have

$$-f_x(x_0,y_0)(x-x_0)-f_y(x_0,y_0)(y-y_0)+(z-z_0)=0$$
$$\Longrightarrow z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

normal line: $\begin{cases}x=x_0-f_x(x_0,y_0)\cdot t\\ y=y_0-f_y(x_0,y_0)\cdot t\\ z=z_0+t\end{cases}$

e.x. $z=f(x,y)=32-3x^2-4y^2$. Find tangent plane and normal line at $(2,1)$

$f_x=-6x$. $f_y=-8y$.

$g(x,y,z)=z-f(x,y)$ ⇒ $\nabla g(x,y,z)=\langle6x,8y,1\rangle$

normal line: $\begin{cases}x=2+12t\\ y=1+8t\\ z=16+t\end{cases}$

tangent plane: $z=16+12(x-2)+8(y-1)$

e.x. Find the parametric equation of the tangent line to the curve of intersection of $z=x^2+y^2$ and $3x^2+2y^2+z^2=9$ at $(1,1,2)$.

$$\begin{cases}g(x,y,z)=z-x^2-y^2=0.\\ h(x,y,z)=3x^2+2y^2+z^2=9\end{cases}$$

$\nabla g(x,y,z)=\langle-2x,-2y,1\rangle$

$\nabla h(x,y,z)=\langle6x,4y,2z\rangle$

let $\vec{u}=\nabla g(1,1,2)=\langle-2,-2,1\rangle$

$\vec{v}=\nabla h(1,1,2)=\langle6,4,4\rangle$

$\vec{u}\times\vec{v}=\langle12,-14,-4\rangle=\langle6,-7,-2\rangle$

∴ tangent line: $\begin{cases}x=1+6t\\ y=1-7t\\ z=2-2t.\end{cases}$

Linear Approximation

$$f(x,y)\approx L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

e.x. $f(x,y)=\frac{5}{x^2+y^2}$ find linear approx at $(-1,2)$

$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

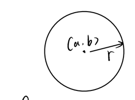
$=1+\frac{10}{25}(x+1)-\frac{20}{25}(y-2)$

$=1+\frac{2}{5}(x+1)-\frac{4}{5}(y-2)$

$L(-1.05,2.1)=1+\frac{2}{5}(-0.05)-\frac{4}{5}(0.1)$

Minimum & Maximum

Open Disk $B((a,b),r)$



Local Extremum $f(x,y)\le f(a,b)$ or $\ge f(a,b)$ for all (x,y) in the domain of $D((a,b),r)$

Absolute Extremum $f(x,y)\le f(a,b)$ or $\ge f(a,b)$ for all (x,y) in the domain of f

if $f(x,y)$ is continous on a closed and bounded domain D. then $f(x,y)$ attains its absolute min/maximum in D

Critical Point

(a,b) is critical point.

if: $f_x(a,b)=f_y(a,b)=0$.

or. at least one of f_x, f_y does not exist at (a,b)

Local extremum $\longrightarrow$ critical point.

$\not\longleftarrow$

$f(x,y)=y^2-x^2$. $f_x(0,0)=f_y(0,0)=0$.

But $(0,0)$ is NOT Local extremum of (x,y)

"Saddle Point"

Second Derivative Test

$$D=\begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b)\\ f_{yx}(a,b) & f_{yy}(a,b)\end{vmatrix}=f_{xx}(a,b)f_{yy}(a,b)-f_{xy}(a,b)^2$$

- if $D>0$ and $f_{xx}(a,b)>0$. ⇒ Local Minimum
- if $D>0$ and $f_{xx}(a,b)<0$. ⇒ Local Maximum
- if $D<0$. saddle point. if. $D=0$. inconclusive

e.x. $f(x,y)=xy(x-2)(x+3)$ Classify all critical points of f

$f(x,y)=(x^2y-2xy)(x+3)$

$=x^3y+3x^2y-2x^2y-6xy$

$f_x=3x^2y+6xy-4xy-6y=y(3x^2+2x-6)$

$f_y=x^3+3x^2-2x^2-6x$

$=x(x^2+x-6)=x(x+3)(x-2)$

$f_y=0$. $x=0$ or $x=-3$ or $x=2$.

when $x=0$. $f_x=-6y$. ⇒ $f_x=0$. $y=0$.

when $x=-3$ $f_x=y(27-12)$ ⇒ $f_x=0$. $y=0$.

when $x=2$ $f_x=y(10)$ ⇒ $f_x=0$. $y=0$.

∴ critical point. $(0,0)$, $(2,0)$, $(-3,0)$.

$$\begin{cases} f_{xx}=y(6x+2)\\ f_{xy}=3x^2+2x-6.\\ f_{yy}=0\end{cases}$$

for $(0,0)$: $D=0-(3x^2+2x-6)^2=-36$

$D<0$. saddle point.

for $(2,0)$: $D=-(3x^2+2x-6)^2=-100<0$.

saddle point.

for $(-3,0)$: $D=-10$. Saddle Point.

e.x. $f(x,y)=xy(x-2)(y+3)$ Classify all critical points of f

$f(x,y)=(x^2y-2xy)(y+3)=x^2y^2+3x^2y-2xy^2-6xy$.

$f_x=2xy^2+6xy-2y^2-6y=2y^2(x-1)+6y(x-1)=2y(y+3)(x-1)$

$f_y=2x^2y+3x^2-4xy-6x=x^2(2y+3)-2x(2y+3)=x(x-2)(2y+3)$

when $f_x=0$. $y=0$ or $y=-3$ or $x=1$.

when $y=0$. $f_x=3x(x-2)$ ⇒ $f_x=0$, $x=0$ or $x=2$

when $y=-3$ $f_x=-3x(x-2)$ ⇒ $f_x=0$. $x=0$ or $x=2$.

when $x=1$ $f_x=-(2y+3)$ ⇒ $f_y=0$ $y=-\frac{3}{2}$.

∴ critical points: $(0,0)$, $(2,0)$, $(0,-3)$, $(2,-3)$, $(1,-\frac{3}{2})$

$f_{xx}=2y(y+3)$ $f_{xy}=(x-1)[4y+6]$ $f_{yy}=x(x-2)\cdot2$.

$$D=\begin{vmatrix} f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{vmatrix}=2y(y+3)\cdot2x(x-2)-(x-1)^2(4x+6)^2$$

$=4xy(x-2)(y+3)-(x-1)^2(4x+6)^2$

for $(0,0)$, $D=-36<0$. saddle point.

for $(2,0)$. $D=0-1\cdot14^2<0$. saddle point.

for $(0,-3)$. $D=-36<0$. saddle point.

for $(2,-3)$ $D<0$. saddle point.

for $(1,-\frac{3}{2})$ $D=4\cdot(-\frac{3}{2})(-1)(\frac{3}{2})>0$, $f_{xx}=2\cdot-\frac{3}{2}\cdot\frac{3}{2}<0$.

∴ Local maximum

Finding Absolute max/minimum

Finding absolute maximum/minimum

Suppose f is continuous on a closed and bounded domain R in $\mathbb{R}^2$. By extremum value theorem, f attains its absolute maximum and minimum in R. If absolute maximum/minimum value is attained in the interior of R i.e. not on the boundary, then it must be a local maximum/minimum, which implies that it is a critical point of f. Therefore, if we search for absolute maximum/minimum, we only need to check the values of f at the critical points and on the boundary of R.

The general procedure for finding absolute maximum/minimum is as follows:

1. Find all critical points in R and determine their values of f.
2. Parametrize the boundary of R and find the maximum and minimum values of f on the boundary.
3. The greatest value found in the first two steps is the absolute maximum value of f on R, and the smallest value found in the first two steps is the absolute minimum value of f on R.

e.x. $f(x,y)=x^2+y^2-2x+2y+5$. $R=\{(x,y)\mid x^2+y^2\le4\}$, find absolute max/min.

1. find all critical points.

$f_x=2x-2$ $f_y=2y+2$.

$f_x=0$. $x=1$.

when $x=1$. $f_y=0$ ⇒ $y=-1$.

critical point: $(1,-1)$ ⇒ $f(1,-1)=3$.

2. Boundary $x^2+y^2=4$. Let $x=2\cos t$. $y=2\sin t$.

boundary: $\vec{r}(t)=\langle2\cos t,2\sin t\rangle$.

$g(t)=f(x(t),y(t))=4-4\cos t+4\sin t+5$.

$=2(\sin t-\cos t)+9=4\sqrt2\sin(t-\frac{\pi}{4})+9$

$$\begin{cases} t=\frac{3\pi}{4}. & g(t)\text{ has maximum }4\sqrt2+9\\ t=-\frac{\pi}{4} & g(t)\text{ has minimum }-4\sqrt2+9.\end{cases}$$

3. Compare f(critical points) and max/min at boundary.

⇒ absolute maximum $9+4\sqrt2$.

absolute minimum 3.

Exercise: Suppose $f(x,y)=3xy-6x-3y+7$ is defined on the domain T, a closed region whose boundary is the triangle with vertices $(0,0),(3,0)$, and $(0,5)$. Let L_1 be the line segment joining $(0,0)$ and $(0,5)$, L_2 be the line segment joining $(0,0)$ and $(3,0)$, and L_3 be the line segment joining $(3,0)$ and $(0,5)$.

(a) Parametrize L_1, L_2, and L_3.
(b) Find the absolute maximum and minimum values of f on T.

(a). $\vec{L}_1(t)=\langle0,t\rangle$ $t\in[0,5]$.

$\vec{L}_2(t)=\langle t,0\rangle$ $t\in[0,3]$

$\vec{L}_3(t)=\langle t,-\frac{5}{3}t+5\rangle$ $t\in[0,3]$.

(b) $f_x=3y-6$ $f_y=3x-3$.

critical point. $(1,2)$, $f(1,2)=6-6-6+7=1$.

for L_1. $g(t)=f(x(t),y(t))=-3t+7$ ($t\in[0,5]$)

$g(t)$ has maximum 7 at $t=0$.

has minimum -8 at $t=5$.

for L_2. $g(t)=f(x(t),y(t))=-6t+7$ ($t\in[0,3]$)

$g(t)$ has maximum 7 at $t=0$.

has minimum -11 at $t=3$

for L_3 $g(t)=3t(5-\frac{5}{3}t)-6t-3(-\frac{5}{3}t+5)+7$

$=-5t^2+14t-8$. ($t\in[0,3]$).

$g(0)=-8$. $g(3)=-11$

$g(\frac{14}{10})=g(\frac{7}{5})=-5\cdot\frac{7}{5}\cdot\frac{7}{5}+14\cdot\frac{7}{5}-8.=-\frac{49}{5}+\frac{98}{5}-\frac{40}{5}=\frac{9}{5}$

∴ Abs maximum 7

Abs minimum -11.