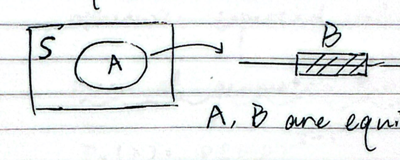
Random Variable.

$\begin{cases} \text{Sample Space} : S \to \text{domain.} \\ \text{All possible value of } X \to \text{Range.} \end{cases}$

Equivalent Event:



A, B are equivalent.

Discrete R.V.

$\begin{cases} \text{probability mass function (pmf)} \\ \text{Cumulative distribution function (cdf)} \\ \text{Characteristic function} \end{cases}$

pmf:

$p_X(x) = P[X = x]$ where: $S_X = \{x_1 \dots x_n\}$.

property:

① $p_X(x) \geq 0$. ② $\sum_x p_X(x) = 1$

③ $P[X \text{ in } B] = \sum_{x \in B} p_X(x)$ if $B \subset S_X$.

Expectation of a random variable.

$m_X = E[X] = \sum_{x \in S_X} x\, p_X(x)$

the expected value is defined

if, $\sum_k |x_k| p_X(x_k) < \infty$.

otherwise expected value doesn't exist

mean of geometric R.v.:

$p_X(x) = p(1-p)^{x-1}$

$E(X) = \sum_{k=1}^{\infty} k \cdot p(1-p)^{k-1}$

$= p \cdot \sum_{k=1}^{\infty} k(1-p)^{k-1}$

$\cancel{= p\sum_{k=0}^{\infty}(k+1)(1-p)^{k}}$

$\cancel{= p \lim_{k\to\infty} \frac{1-(1-p)^k}{p}}$

$\cancel{[(1-p)^k]' = k(1-p)^{k-1}}$

for $\sum_{k=1}^{\infty} k(1-p)^{k-1}$

let $S = \sum_{k=1}^{\infty} k(1-p)^{k-1} = \sum_{k=1}^{\infty} k x^{k-1}$.

$= 1 + 2x + 3x^2 \dots$

$xS = \quad x + 2x^2 + \dots$

$(1-x)S = 1 + x + x^2 \dots = \sum_{i=0}^{\infty} x^i = \lim_{i\to\infty} \frac{1-(1-p)^i}{1-(1-p)}$

$\Rightarrow S = \frac{1}{p} \cdot \frac{1}{1-(1-p)} = \frac{1}{p^2} \qquad = \frac{1}{p}$

$\therefore E(X) = \sum_{k=1}^{\infty} p\,k(1-p)^{k-1}$

$= p \cdot \frac{1}{p^2} = \frac{1}{p}$

Mean of Binomial R.V.

$E(X) = \sum_{k=0}^{n} k \cdot p_X(k)$

$= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$.

$= \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$.

$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k}$

$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} p^{k-1} (1-p)^{n-k}$

$= np \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} p^j (1-p)^{n-1-j}$

$= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j}$

$= np$

Expectation of a function of R.v.

$E(g(X)) = \sum_k g(k) p_X(k)$

useful results:

$\begin{cases} E[aX] = aE[X] \\ E[c] = c. \text{ where } c \text{ is a constant.} \\ E[X+c] = E[X] + c. \\ E[ag(X)] = aE[g(X)] \end{cases}$

let X be r.v. $S_X = \{-3, -1, 1, 3\}$. with equal prob $\frac{1}{4}$.

Find $E(Z)$, $Z = (2X+10)^2$.

$Z = (2X+10)^2 = 4X^2 + 40X + 100$.

$E(X) = \sum_{i \in S_X} i \cdot p_X(i)$
$= [-3 + (-1) + 1 + 3] \frac{1}{4} = 0$.

$E(X^2) = [9+1+1+9] \frac{1}{4} = 5$.

$E(Z) = 4E(X^2) + 40E(X) + 100$.
$= 120$.

Variance of an R.v.

$\begin{cases} Var(X) = E[(X - E(X))^2] \\ Std(X) = \sqrt{Var(X)} \end{cases}$

Property of $Var(X)$

$\begin{cases} Var(X) = [E(X^2)] - [E(X)]^2 \\ Var(c) = 0. \\ Var(X+c) = Var(X) \\ Var(cX) = c^2 Var(X) \end{cases}$

Moments of a random variable.

n^{th} moment:

$E(X^n) = \sum_k x_k^n p_X(x_k)$

n^{th} central moment:

$E[[X - E(X)]^n]$

$= \sum_k (x_k - E(X))^n p_X(x_k)$

Important Discrete R.v.

Bernoulli R.v.

$p_X(0) = 1-p \quad p_X(1) = p$.

$E(X) = 0 \cdot (1-p) + 1 \cdot p = p$.

$Var(X) = \cancel{[E(X)]^2 - E(X^2)}$

$= E[X^2] - [E(X)]^2$

$= p - p^2 = p(1-p)$

Binomial R.v.

$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$

$= np$.

$Var(X) = E(X^2) - [E(X)]^2$

$=$

$E(X(X-1)) = \sum_{k=0}^{n} (k)(k-1) \binom{n}{k} p^k (1-p)^{n-k}$

$= \sum_{k=2}^{n} k(k-1) \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$

$= \sum \frac{(n-2)!}{(k-2)!(n-2-(k-2))!} p^{k-2} (1-p)^{(n-2)-(k-2)}$

$= n(n-1)p^2 \sum_{j=0}^{n-2} \frac{(n-2)!}{j!(n-2-j)!} p^j (1-p)^{(n-2)-j}$

$= n(n-1)p^2$.

$\Rightarrow E(X^2) - E(X) = n(n-1)p^2$

$E(X^2) = n(n-1)p^2 + np$.

$Var(X) = E(X^2) - [E(X)]^2$

$= n(n-1)p^2 + np - n^2p^2$

$= n^2p^2 - np^2 + np - n^2p^2$

$= np(1-p)$

Geometric R.v.

$E(X) = \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \cdot p$

$= \frac{1}{p}$

$E(X^2) = \sum_{i=1}^{\infty} i^2 p(1-p)^{i-1}$

$\cancel{(\sum_{i=1}^{\infty} i) \cdot p \quad \frac{1}{p^2}}$

$\cancel{\sum_{i=1}^{\infty} i(1-p)^{i-1}}$

$\cancel{S = \sum_{i=1}^{\infty} i x^{i-1}}$

$\cancel{S = 1 + 4x + 9x^2 \dots}$

$E(X(X-1)) = \sum_{k=1}^{\infty} k(k-1) p \cdot (1-p)^{k-1}$

$= p \sum_{k=1}^{\infty} k(k-1)(1-p)^{k-1}$

$= p \cdot \frac{2(1-p)}{p^3} = \frac{2(1-p)}{p^2}$

$\cancel{\sum_k (1-p)^k = \frac{1}{p}}$

$\cancel{\sum_k k(1-p)^{k-1} \quad \frac{1}{p^2}}$

$\frac{-2}{p^3}$

$E(X^2) - E(X) = \frac{2(1-p)}{p^2}$

$E(X^2) = \frac{2(1-p)}{p^2} + \frac{1}{p} = \frac{2-p}{p^2}$

$Var(X) = E(X^2) - [E(X)]^2$

$= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$

Memoryless property of Geometric R.v.

$P[M = k+m \mid M > m]$

$= \frac{P[M = k+m \cap M > m]}{P[M > m]}$

$= \frac{(1-p)^{k+m-1} \cdot p}{(1-p)^m} = (1-p)^{k-1} \cdot p$

$= P[M = k]$

Memoryless: since independence of bernolli trial

Discrete Uniform R.v.

$p_X(k) = \frac{1}{M}$ for $k = 1 \dots M$.

$E(X) = \frac{M-1}{2}$

$Var(X) = E(X^2) - [E(X)]^2 = \frac{M^2-1}{12}$.

Poisson R.v.

occurences of an event in a certain interval of space & time.

$P[N = k] = \frac{\alpha^k}{k!} e^{-\alpha}$.

where α is the average number of events in the interval.

$\begin{cases} E[N] = \alpha. \\ Var[N] = \alpha. \end{cases}$

remarks:

① $\sum_{k=-\infty}^{\infty} p_X(k) = 1$.

since # occurences is positive int.

$\sum_{k=-\infty}^{\infty} p_X(k) = \sum_{k=0}^{\infty} p_X(k)$

$= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} e^{-\alpha}$

$= e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} = e^{-\alpha} \cdot e^{\alpha} = 1$

② $P[N = k]$ has maximum on $k = \lfloor \alpha \rfloor$

i.e. the largest integer smaller than α.

③ for large n and small p.

the pmf of binomial RV

can be approximated by poisson RV.

$\binom{n}{p} p^k (1-p)^{n-k} = \frac{\alpha^k}{k!} e^{-\alpha}$.

proof:

For binomial R.V. with $p = \alpha/n$.

p_0: ~~prof~~ prob of no success.

$p_0 = (1-p)^n = (1 - \frac{\alpha}{n})^n = e^{-\alpha}$
(as $n \to \infty$)

$(\lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x)$

$\frac{p_{k+1}}{p_k} = \frac{\binom{n}{k+1} p^{k+1} (1-p)^{n-k-1}}{\binom{n}{k} p^k (1-p)^{n-k}}$

$= \frac{p}{1-p} \cdot \frac{\frac{n!}{(k+1)!(n-k-1)!}}{\frac{n!}{k!(n-k)!}}$

$= \frac{p}{1-p} \cdot \frac{(n-k)}{(k+1)}$

$= \frac{n-k}{k+1} \cdot \frac{\frac{\alpha}{n}}{1 - \frac{\alpha}{n}}$

$= \frac{(1 - \frac{k}{n})\alpha}{(k+1)(1 - \frac{\alpha}{n})}$

when $n \to \infty$

$\frac{p_{k+1}}{p_k} = \frac{\alpha}{k+1}$

$p_k = \frac{\alpha^k}{k!} e^{-\alpha} \Leftarrow p_0 = e^{-\alpha} \quad p_{k+1} = \frac{\alpha}{k+1} p_k$.

$p_k = \cancel{\frac{\alpha}{k+1}} \cdot \frac{\alpha}{k} \cdot \frac{\alpha}{(k-1)} \cdots \frac{\alpha}{1} \cdot e^{-\alpha}$

Continuous R.V.

$\begin{cases} \text{Cumulative distribution function} \\ \text{Probability density function} \\ \text{Characteristic function} \end{cases}$

Cdf: $F_X(x) = P[X \leq x]$

Unit Step function

$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$

For discrete R.v.

$F_X(x) =$
$p_X(x_1) u(x - x_1)$
$+ p_X(x_2) u(x - x_2)$
$+ p_X(x_3) u(x - x_3)$
$\dots$

property of cdf:

$\begin{cases} 0 \leq F_X(x) \leq 1. \\ \lim_{x\to\infty} F_X(x) = 1 \\ \lim_{x\to-\infty} F_X(x) = 0 \end{cases}$

$F_X(x)$ is non-decreasing.

Pdf:

$f_X(x) = \frac{dF(x)}{dx}$

$f_X(x) \geq 0$.

$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$.

$\int_{-\infty}^{\infty} f_X(t)\,dt = 1$.

$P[a < X \leq b] = F_X(b) - F_X(a^+)$

$= \int_{a^+}^{b} f_X(t)\,dt$.

pdf of discrete R.v.

$f_X(x) = \sum_k p_k \delta(x - x_k)$

Expectation of continuous R.v.

$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$.

only exists if $\int_{-\infty}^{\infty} |t| f_X(t)\,dt < \infty$

Mean of expotential RV.

$f_X = \lambda e^{-\lambda x} u(x)$

$E(X) = \int_{-\infty}^{\infty} x \lambda e^{-\lambda x} u(x)\,dx$

$= \int_0^{\infty} \lambda e^{-\lambda x} x\,dx$.

$= \frac{1}{\lambda}$

$\begin{cases} \text{pmf: Probability mass function: } p_X(x) \\ \text{pdf: Probability density function: } f_X(x) \\ \text{cdf: Cumulative distribution function: } F_X(x) \end{cases}$

Linearity of expectation

$E[\sum_i a_i g_i(X)] = \sum_i a_i E[g_i(X)]$

Variance of Continuous R.v.

① Var of Uniform RV

$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise.} \end{cases}$

$E[X] = \int_a^b \frac{x}{b-a}\,dx = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$

$E[X^2] = \int_a^b \frac{x^2}{b-a}\,dx = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + ab + b^2}{3}$

$Var[X] = E[X^2] - (E[X])^2 = \frac{a^2 + ab + b^2}{3} - \frac{a^2 + 2ab + b^2}{4} = \frac{a^2 - 2ab + b^2}{12} = \frac{(a-b)^2}{12}$

MSE: Mean Square Error

How "good" is an estimate "c" of a RV X

$MSE = E[(X-c)^2] = \int_{-\infty}^{\infty} (x-c)^2 f_X(x)\,dx$

$\frac{d}{dc} MSE = -\int_{-\infty}^{\infty} 2(x-c) f_X(x)\,dx$

$= -2\int_{-\infty}^{\infty} x f_X(x)\,dx + 2c\int_{-\infty}^{\infty} f_X(x)\,dx$

$= 2c - 2E(X)$

when minimize MSE: $\frac{d}{dc} MSE = 0 \Rightarrow c = E(X)$

Important Continuous RV

① Uniform RV

$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$

$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x}{b-a}, & a \leq x \leq b \\ 1, & x > b \end{cases}$

$E[X] = \frac{a+b}{2}$, $Var[X] = \frac{(a-b)^2}{12}$

② Expotential RV "Waiting Time"

$f_X(x) \begin{cases} 0, & x < 0. \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases}$

$F_X(x) = \int_0^x \lambda e^{-\lambda x}\,dx = 1 - e^{-\lambda x}$ for $x \geq 0$.

$F_X(x) = \begin{cases} 0, & x < 0. \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases} \qquad -\lambda \cdot e^{-\lambda x}$

$E[X] = \int_0^{\infty} \lambda x e^{-\lambda x}\,dx = \int_0^{\infty} -x\,d e^{-\lambda x} = -xe^{-\lambda x}\Big|_0^{\infty} - \int_0^{\infty} -e^{-\lambda x}\,dx$

$= -xe^{-\lambda x}\Big|_0^{x} - \frac{1}{\lambda} e^{-\lambda x}\Big|_0^{\infty}$

$= \frac{1}{\lambda}$

$Var[X] = \frac{1}{\lambda^2}$

Memoryless Property of Exponential RV

$P[X > t+h \mid X > t] = P[X > h]$

③ Gaussian RV

$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$

$\Rightarrow$ mean: m. Std: σ

cdf: $F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx$.

for normalized gaussian RV.

$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}}\,dx$

$F_X(x) = \Phi(\frac{x-m}{\sigma})$

Q-Function: $Q(x) = 1 - \Phi(x)$

$Q(x) = \Phi(-x) = 1 - \Phi(x)$

Conditional PMF / CDF / PDF

Conditional Pmf

$p_X(x \mid C) = \frac{P(C \cap \{X = x\})}{P(C)}$

- In many instances, the conditioning event, C, is defined in terms of X
  - E.g. C = "$X > 10$" or C = "$a \leq X \leq b$"
- In this case, the conditional pmf can be determined **solely from the pmf of X.**
- For x_k in S_X, $p_X(x_k \mid C) = \frac{P[\{X = x_k\} \cap C]}{P[C]} = \begin{cases} \frac{p_X(x_k)}{\sum_{x \in C} p_X(x)} & \text{if } x_k \in C \\ 0 & \text{if } x_k \notin C \end{cases}$ From PMF to conditional PMF

Total Probability Theorem

$p_X(x) = \sum_i p_X(x \mid B_i) P(B_i)$

- In cases where it is natural to analyze the outcome of an experiment by *conditioning on individual events* $B_1, B_2, \dots B_n$ in a partition of a sample space, we can **compute expected values from conditional expected values**:

$$\boxed{E[X] = \sum_i E[X \mid B_i] P[B_i]}$$

Proof

$$E[X] = \sum_k x_k p_X(x_k)$$
$$= \sum_k x_k \left\{ \sum_i p_X(x_k \mid B_i) P[B_i] \right\}$$
$$= \sum_i \left\{ \sum_k x_k p_X(x_k \mid B_i) \right\} P[B_i]$$
$$= \sum_i E[X \mid B_i] P[B_i]$$

- We can compute the average either by summing the observations in order, or by counting the number of times each outcome occurs.

$$\langle X \rangle_n = \frac{x(1) + x(2) + \dots + x(n)}{n} = \frac{x_1 N_1(n) + x_2 N_2(n) + \dots + x_k N_k(n) + \cdots}{n}$$
$$= x_1 \frac{N_1(n)}{n} + x_2 \frac{N_2(n)}{n} + \dots + x_k \frac{N_k(n)}{n} + \cdots$$
$$= x_1 f_1(n) + x_2 f_k(n) + \dots + x_k f_k(n) + \dots = \sum_k x_k f_k(n)$$

- From the previous page, the average value of X in n trials is

$$\boxed{\langle X \rangle_n = \sum_k x_k f_k(n)}$$

- As the number of trials, n, increases, $\lim_{n\to\infty} f_k(n) = p_X(x_k)$ for all k
- This implies that

$$\boxed{\langle X \rangle_n = \sum_k x_k f_k(n) \xrightarrow[n\to\infty]{} \sum_k x_k p_X(x_k) = E[X]}$$

Function of a Random Variable

① if Y is discrete.

$p_Y(k) = P[Y = k] = P[\{X : g(X) = k\}]$

② if Y is continuous:

$\Rightarrow F_Y(y) = P[Y \leq y] = P[\{X : g(X) \leq y\}]$

$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy} = \sum_{x: g(x) = y} \frac{f_X(x)}{|g'(x)|}$