Basic Probability Theorem

Lecture 1

① Model $\xrightarrow{\text{explain}}$ Behaviour

Mathmatical / Computer simulation / Deterministic / Probabilistic

Probabilistic Model

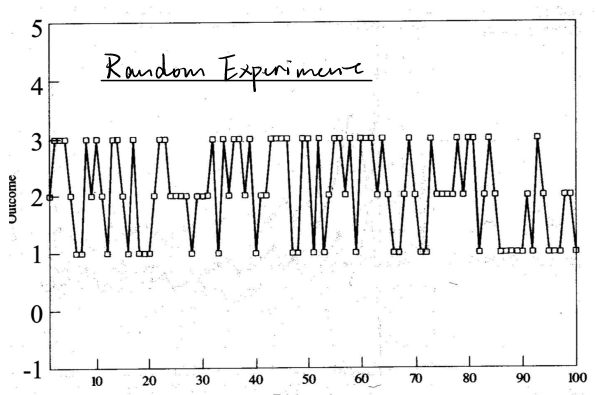
if A $\xrightarrow{P_1}$ B will happen

if A $\xrightarrow{P_2}$ C will happen

② Relative Frequency

e.x. urn, 3 balls labelled 1, 2, 3.

⇒ Selected one → recorded → return



$N_k(n)$: # times we pick ball k in n trials

Relative Frequency: $f_k(n) = \frac{N_k(n)}{n}$

Probability $\boxed{p_k = \lim_{n\to\infty} f_k(n)}$

Property of relative frequency

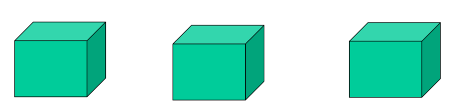
① $0 \le f_k(n) \le 1$ since $N_k(n) \in [0, n]$

② $\sum_k f_k(n) = 1$

③ if $C = \{A \text{ or } B\}$ $f_C(n) = f_A(n) + f_B(n)$

Example on Probability:

Suppose exactly 1 out of 3 boxes contains an **iPad**, and the remaining two boxes contain a homework



① Pick one randomly

② Host take away one of boxes I didn't select. which contains homework.

③ Will I change my choice?

Before taken away: $p(\text{ipad}) = \frac{1}{3}$.

After: If not change: still $\frac{1}{3}$

If change: $\frac{1}{3}$: ipad $\xrightarrow{\text{swap}}$ homework,

$\frac{2}{3}$: homework $\xrightarrow{\text{swap}}$ ipad

∴ $p(\text{ipad}) = \frac{2}{3}$, higher ⇒ I should change my choice!

Lecture 2

① random experiment: 1. procedure. 2. measurement/observation

② Sample space S: set of all possible outcomes.

(Discrete / Continous)

③ Event A: $A \subseteq S$: (Subset of S)

- *Experiment 1*: Select a ball from an urn containing balls labeled 1 to 50. Note the number on the ball.
  $S_1 = \{1, 2, 3, 4, \ldots\ldots, 49, 50\}$
  A_1 = "An even numbered ball is selected" = $\{2, 4, 6, \ldots, 48, 50\}$
- *Experiment 2*: Select a ball from an urn containing balls labeled 1 to 50. Note the number in the tens place.
  $S_2 = \{0, 1, 2, 3, 4, 5\}$
  A_2 = "The number is more than 2" $\{3, 4, 5\}$
- *Experiment 3a*: Toss a coin in the air and note which side faces up when it lands. The two sides of a coin are often called "heads" (H) and "tails" (T).
  $S_{3a} = \{H, T\}$
  A_{3a} = "The coin lands with heads up" = $\{H\}$
- *Experiment 3b*: Toss a coin three times. Note the sequence of heads/tails.
  $S_{3b} = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$
  A_{3b} = "The three tosses give the same outcome" = $\{HHH, TTT\}$

Set Theory

operation: $A \cup B$, $A \cap B$, A^c ($\bar{A}$)

relationship: $A = B$, $A \subset B$, $A \cap B = \phi$ (mutually exclusive)

Commutative Properties:

$$A \cup B = B \cup A \text{ and } A \cap B = B \cap A$$

Associative Properties:

$$A \cup (B \cup C) = (A \cup B) \cup C \text{ and } A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive Properties:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \text{ and}$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

DeMorgan's Rule:

$$(A \cap B)^c = A^c \cup B^c \text{ and } (A \cup B)^c = A^c \cap B^c$$

Three axiom of Probability

Axiom 1: $0 \le P[A]$

Axiom 2: $P[S] = 1$

Axiom 3: if $A \cap B = \phi$, then $P[A \cup B] = P[A] + P[B]$

General: $A_1, A_2 \ldots A_n$. $A_i \cap A_j = \phi$ for all $i \ne j$ ⇒ $P[\sum_{k=1}^{n} A_k] = \sum_{i=1}^{n} P[A_k]$

Collary:

① $P[A^c] = 1 - P[A]$

proof: $A + A^c = S$. $A \cap A^c = \phi$.

⇒ $P[A \cup A^c] = P[A] + P[A^c] = P[S] = 1$.

∴ $P[A^c] = 1 - P[A]$

② $P[A] \le 1$

③ $P[\phi] = 0$.

④ If $A_1, A_2, \ldots. A_n$ are pair-wise mutually exclusive, then $P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{k=1}^{n} P[A_k]$ for $n \ge 2$

⑤ $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

⑥ $A \subset B \Rightarrow P[A] \le P[B]$

- The axioms and corollaries provide with a set of rules for computing probabilities of events in terms of other events (*mathematical manipulation*).
- Note that any assignment of probabilities to events that satisfies the three axioms can be manipulated in this way. It is up to the **engineer** to determine a "reasonable" or "accurate" assignment of probabilities.
- Assigning probabilities to some **sub-class of events** gives you enough information to determine the probability of any event (by manipulating the axioms and corollaries).

Lecture 3:

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$P(A)$: prior

$P(A|B)$: posterior

property of Conditional Probability

① if $B \subset A$. $P(A|B) = 1$

② if $A \subset B$ then $P(A|B) = \frac{P(A)}{P(B)} \ge P(A)$

③ for fixed B, $P(A|B)$ is a probability measure.

Proof:

Axiom I: $P[A|B] \ge 0$

Proof $P[A \cap B] \ge 0$ and $P[B] > 0 \Rightarrow P[A|B] = \frac{P[A \cap B]}{P[B]} \ge 0$

Axiom II: $P[S|B] = 1$

Proof $S \cap B = B$, $P[S|B] = \frac{P[S \cap B]}{P[B]} = \frac{P[B]}{P[B]} = 1$

Axiom III: If A_1 and A_2 are mutually exclusive, then $P[A_1 \cup A_2 | B] = P[A_1|B] + P[A_2|B]$

Proof $P[A_1 \cup A_2 | B] = \frac{P[(A_1 \cup A_2) \cap B]}{P[B]} = \frac{P[A_1B \cup A_2B]}{P[B]}$

Since A_1 and A_2 are mutually exclusive, so are A_1B and A_2B, therefore $P[A_1 \cup A_2 | B] = \frac{P[A_1B]}{P[B]} + \frac{P[A_2B]}{P[B]} = P[A_1|B] + P[A_2|B]$

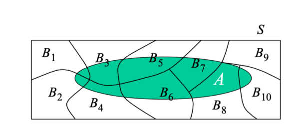
Joint Probability:

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Total Probability:

If $B_1, B_2, B_3, \ldots\ldots, B_n$ partition the sample space, then for any event A,

$$P[A] = \sum_{i=1}^{n} P[A|B_i]P[B_i]$$



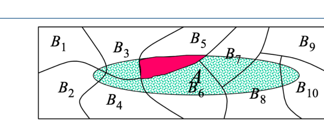
Proof:

$$A = A \cap S = A \cap \left(\bigcup_{i=1}^{n} B_i\right) = \bigcup_{i=1}^{n} AB_i$$

where $AB_1, AB_2, \ldots, AB_n$ are mutually exclusive.

- By Corollary 4, $P[A] = \sum_{i=1}^{n} P[AB_i] = \sum_{i=1}^{n} P[A|B_i]P[B_i]$

Bayes' rule



Let $B_1, B_2, B_3, \ldots, B_n$ be a partition of the sample space S. For any event A,

$$P[B_j|A] = \frac{P[AB_j]}{P[A]} = \frac{P[A|B_j]P[B_j]}{\sum_{k=1}^{n} P[A|B_k]P[B_k]}$$

Independence

A, B are independent if $P[A] \cdot P[B] = P[A \cap B]$

⇒ $P[A|B] = P[A]$ $P[B|A] = P[B]$

Medical Testing

Sensitivity = P(test + | has disease)

$= \frac{TP}{TP + FN}$

Specificity = P(test - | not has disease).

$= \frac{TN}{TN + FP}$

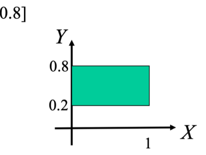
Lecture 4: Sequential Experiments

experiments w/ repetition on multiple participants

Sample space: Cartesian product of individuals sample space.

$\{H, T\} \times \{H, T\} = \{HH, HT, TH, TT\}$

- Example: Pick a random number X from $[0, 1]$ and a random number Y from $[0.2, 0.8]$. The sample space is $[0, 1] \times [0.2, 0.8]$



Bernoulli trails

trail $\xrightarrow{P}$ success

trail $\xrightarrow{1-P}$ failure.

Bernolli Probability law.

let k: # successes $P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}$

⇒ $\sum_{k=0}^{n} P_n(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$.

Can be proved by bionomial theorem

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ (let $a = p$, $b = 1-p$)

Multinomial Probability law

$B_1 \ldots B_M$ partition the sample space

let $P(B_j) = p_j$ ⇒ $p_1 + \cdots + p_M = 1$

k_j: # times event B_j occurs

$$\boxed{P[(k_1, k_2, \ldots, k_M)] = \frac{n!}{k_1! \cdots k_M!} p_1^{k_1} p_2^{k_2} \cdots p_M^{k_M}}$$

Geometric Probability law

We conduct an independent sequence of Bernoulli trials

M: # trials until first success

$P(M = m) = p(1-p)^{m-1}$

N: # trials Before first success.

$P(N = n) = p(1-p)^n$

① $\sum_{m=1}^{\infty} P(m) = \sum_{m=1}^{\infty} p(1-p)^{m-1}$

$= p \sum_{m'=0}^{\infty} (1-p)^{m'}$

$= p \cdot \frac{1}{1-(1-p)} = 1$.

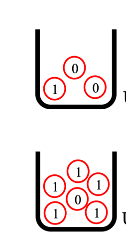
② $P(M > k) = \sum_{m=k+1}^{\infty} p(1-p)^{m-1}$

$= p(1-p)^k \sum_{m'=0}^{\infty} (1-p)^{m'}$

$= (1-p)^k$

Sequence of dependent experiments

Find the probability of the urn sequence 0011 for the urn experiment introduced in the previous example.

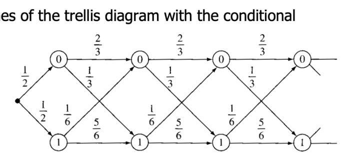


Urn 0

Urn 1

Solution:

We can label the branches of the trellis diagram with the conditional probabilities:



Using the conditional probability, we obtain

$$P[0011] = P[1|1] \times P[1|0] \times P[0|0] \times P[0]$$
$$= \frac{5}{6} \times \frac{1}{3} \times \frac{2}{3} \times \frac{1}{2} = \frac{5}{54}$$