Auctions

Widely used in Consumer/Corporate/CS settings...

Provide a fundamental theoretical framework for understanding resource allocation problems among self-interested agents

An auction is any protocol that allows agents to indicate their interest in one or more resources, and that uses these indications of interest to determine both an allocation of resources and a set of payments by the agents.

Auctions as Structured Negotiations

- Bidding rules: How are offers made (By whom? When? What can there content be?)
- Clearing rules: When do trade occurs or what are those trades
- Information rules: Who knows what when about the state of negotiation

First Price Auction:

- Bidders send in their bids sealed.
- The highest bidder wins the item at the price of the highest bid.
- When there is a tie, break it randomly.

Second Price Auction:

- Bidders send in their bids sealed.
- The highest bidder wins the item at the price of the second highest bid.
- When there is a tie, break it randomly.

Auctions as Game

① First-Price Auctions

$N = \{1, \dots, n\}$,
$A = \{x \mid x \in [0,1]\}$,
$u_i(x_1, \dots, x_n) = \begin{cases} v_i - x_i & \text{if agent } i \text{ wins} \\ 0 & \text{otherwise} \end{cases}$

② Second Price Auctions

$N = \{1, \dots, n\}$,
$A = \{x \mid x \in [0,1]\}$,
$u_i(x_1, \dots, x_n) = \begin{cases} v_i - x_j & \text{if agent } i \text{ wins} \\ 0 & \text{otherwise} \end{cases}$

☆ Where x_j is the second highest bid

First Price Auction with Common Same Value

Assumptions:

① One single item up for auction using the first price mechanism
② Ties are broken randomly.
③ n bidders, bid using price from fixed set
④ each has value 1 for each item

$u = \begin{cases} v_i - x_i = 1 - x_i, & \text{if } i \text{ wins} \\ 0, & \text{otherwise} \end{cases}$

2-Bidder Case

⇒ $N = \{1, 2\}$, $P = \{0, 1\}$.

u:

	0	1
0	(0.5, 0.5)	(0, 0)
1	(0, 0)	(0, 0)

Nash Equilibrium: (0, 0), (1, 1)

General Case: $P = \{a, b\}$ $0 \le a < b \le 1$

$N = \{1, 2\}$, $P = \{a, b\}$.

u:

1 \ 2	a	b
a	$(\frac{1-a}{2}, \frac{1-a}{2})$	$(0, 1-b)$
b	$(1-b, 0)$	$(\frac{1-b}{2}, \frac{1-b}{2})$

① (b, b) will always be nash equilibrium

② if (a, b) is NE: then $b = 1$ & $\frac{1-a}{2} \le 1-b$

$\frac{1-a}{2} \le 0 \Rightarrow a = 1 \Rightarrow a = b \le 1$ which is impossible

⇒ (a, b) cannot be Nash Equilibrium

③ if (a, a) is NE: $\begin{cases} \frac{1-a}{2} \ge 1-b \\ \frac{1-a}{2} \ge 0 \end{cases} \Rightarrow \begin{cases} b \ge \frac{1+a}{2} \\ a \le 1 \end{cases}$

④ if (b, b) is the ONLY NE: $\frac{1-a}{2} < 1-b$

if More than 2 Bidders: (Discrete Case)

$P = \{a_1, a_2, \dots, a_m\}$, $0 \le a_1 \le a_2 \dots \le a_m \le 1$. $m \ge 2$

① ⇒ $(a_m, a_m, \dots, a_m)$ is always a NE

Proof: for any player j, $s_j = (a_m \dots a_m)$

$\begin{cases} u_j(s_j, a_m) = \frac{1}{n}(1 - a_m) \ge 0 \\ u_j(s_j, a_n)\ (n \ne m) = 0 \end{cases}$ ⇒ Best action in s_j is always a_m for all players.

② $(a_i \dots a_i)$ is a NE iff: $\frac{(1-a_i)}{n} \ge 1 - a_{i+1}$

proof: $(a_i, a_i, \dots, a_i)$ is a NE

⇔ ∀ player j, $s_j = (a_i, a_i, \dots, a_i)$

$\begin{cases} \text{Case 1: } u_j(s_j, a_k)\ (k < i) = 0 \\ \text{Case 2: } u_j(s_j, a_i) = \frac{(1-a_i)}{n} \ge 0 \\ \text{Case 3: } u_j(s_j, a_t)\ (t > i) = 1 - a_t \end{cases}$

⇒ iff $\frac{(1-a_i)}{n} \ge 1 - a_t$, Case 2 is best response, otherwise Case 3

⇒ iff $\frac{(1-a_i)}{n} \ge 1 - a_{i+1}$, NE

③ Consider a profile $x = (x_1, \dots, x_n)$ at least 2 different elements

Largest x is x_i, which occurs k times.

① k = 1 then No NE, proof:

$\begin{cases} \text{if } a_i = 1. \text{ then the bidder bids at } a_i \text{ can lower its bid to get larger payoff (e.g. } 0 \to 1 - a_{i-1}) \\ \text{if } a_i < 1. \text{ then others can raise bids to } a_i \text{ (payoff: } 0 \to \frac{(1-a_i)}{n}) \end{cases}$

② if k > 1. NE iff $a_i = 1$

proof: if $a_i < 1$, then those who bids lower then a_i is not best response (should bid at a_i)

if $a_i = 1$. → players who bid at 1 ⇒ u = 0 ⇒ if lower bid, still 0. (Best response)
↘ players who not. ⇒ u = 0 ⇒ if increase bid. Still 0. (Best response)

⇒ Nash Equilibrium

Continuous Case:

① each player can bid in an interval [l, m] $(0 \le l < m \le 1)$

if m < 1, NE: (m, m, ..., m)

if m = 1, x is NE if more than two "1" occurs in X

② each player can bid in an interval [l, m)

then no NE (since $a_i \to m$)

Auctions as Bayesian Game

Limitations of Auctions as Game

⇒ What the item is worth to an agent is its own private information

Formulation

$N = \{0, 1, 2, \dots, n\}$ (n players)

For each agents. a set of possible privates $v_i \in [a, b]$

a set of possible bids $b_i \in [c, d]$

Common joint prior on private value profiles

$p: [0,1]^n \to [0,1]$ p(v): value profile being $v = (v_1, v_2, \dots, v_n)$.

Payment Functions:

$\sigma_i: [0,1]^n \to [0,1]$, $\sigma_i(b)$: what agent i need to pay when the bid profile is b

Winner Selection: $\tau: [0,1]^n \to N$ $\tau(b)$ is winning agent

$f_i(v_i) \longrightarrow$ agent i's bid b_i "Strategy"

$u_i(f_1, f_2, \dots, f_n) = \sum_v p(v)\, u_i((f_1(v_1), f_2(v_2) \cdots f_n(v_n)), v)$

where $u_i(b, v) = \begin{cases} v_i - \sigma_i(b), & \text{if } \tau(b) = i \\ -\sigma_i(b), & \text{otherwise} \end{cases}$

Nash Equibruim

first-price auction: $(v_1(n-1)/n, v_2(n-1)/n, \dots, v_n(n-1)/n)$

second-price auction: $(v_1, v_2, \dots, v_n)$

Revenue Equivalence Theorem

Any auction in which:

- the bidder with the highest valuation always wins,
- the bidder with the lowest possible valuation expects zero payoff,
- all bidders are risk neutral,
- bidders' valuations are drawn from a strictly increasing and atomless (continuous) distribution,

will lead to the same expected revenue for the seller (auctioneer).

Iterated Prisoner's Dilemma

	Defect	Cooperate
Defect	P=1 P=1	T=5 S=0
Cooperate	S=0 T=5	R=3 R=3

"T > R > P > S"

iterated prisoner dilemma: "Can learn from past experience"

Assumption: 2R > T + S Both Coorapte is better then exploit each other

if finite number of time ⇒ Dominant Strategy & NE: Both Defect

proof: last turn: one might defect ⇒ another cannot retaliate since last turn ⇒ another defect.

⇒ last turn: both defect

↓ second-to-last ⇒ Same logic ---→

Tit-For-Tat: Cooperate at first. then plays whatever move the other player last time

Discount Factor: w. (w ∈ [0, 1])

Round	1	2	3	4	5	...
payoff	s	ws	w^2s	w^3s	w^4s	...

Proposition 1: if w is sufficiently large, there is no best strategy

A strategy is nice if it is not the first to defect

A strategy is **collectively stable** if no strategy can invade it. A strategy A can invade strategy B if given a large group of individuals all using B, a new individual using A can get a better score than the individuals using B.

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Proposition 3

ALL Defect is always collectively stable. However, *ALL Defect* can be invaded by a cluster of individuals, and this cluster needs only be a small proposition of the original population.

Proposition 4

If a nice strategy cannot be invaded by an individual, then it cannot be invaded by any cluster of the original population.