Game theory

Multi-Agent System (MAS).

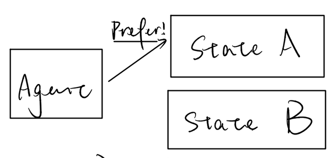
issues:

- ALL issues in Single-Agent System
- How agents communicate / cooperate with each other,
- How they act in face of adversity

Game theory: How self-interested agents interact.
("only care themselves")

Modeling Agents' Interests

Preference:



Agent prefers A over B: $A \succsim B$

How to decide which to prefer?

Utility: Satisfaction of an agent from a certain state of the World

if $u(A) = 100$, $u(B) = 60 \Rightarrow u(A) > u(B) \Rightarrow A \succsim B$.

Properties:

1. for every pair (A, B), either $A \succsim B$ or $B \succsim A$ or Both (Completeness)
2. $(A \succsim B) \wedge (B \succsim C) \rightarrow A \succsim C$ (Transitivity)
3. The desirability of an element in outcome set O does not come from another element's presence
4. Decomposability: For nonempty outcome sets O1 and O2, the most preferred outcome in O1 + O2 is either the most preferred outcome in O1 or the most preferred outcome in O2.
5. Monotonicity: The amount of some elements in A are more than those in B, and no less of any other elements, then A ≽ B. (Preference in economics)
6. Continuity: If A ≽ B, and C is “sufficiently close” to A, then C ≽ B

Theorem: if a preference relation $\succsim$ satisfies all properties above.

Then there exists a function u mapping from the outcome set to a real number in $[0, 1]$

($u: O \rightarrow [0, 1]$), with the property

1. $u(o_1) \geq u(o_2)$ iff $o_1 \succsim o_2$.
2. $u([p_1 : o_1 \ldots p_k : o_k]) = \sum_i p_i u(o_k)$

Game in normal forms

A finite n-player game is a tuple (N, A, u).

- $N = \{1, 2, \ldots n\}$. is a set of players. $i \rightarrow$ player i
- $A: \{A_1, \ldots A_n\} \ni A_i = \{a_i^1, a_i^2 \ldots\}$. (set of actions).
- $u = (u_1 \ldots u_n)$, utility function for agent i. $u_i: A \rightarrow \mathbb{R}$

Prisoner's Dilemma

Two prisoners

- if both deny ⟶ each 1 years.
- if both confess ⟶ each 5 years.
- if one confess (A), one deny (B):
  A is free. B gets 20 years.

Suppose: value: free → 5. 1-year → 3. 5-years → 1 20-year → 0

Formulation:

$N = \{1, 2\}$.

$A = \{A_1, A_2\}$. $A_1 = A_2 = \{$deny, confess$\} = \{C, D\}$.

$u = \{u_1, u_2\}$ ⟶

$A_1 \backslash A_2$	C	D
C	1, 1	5, 0
D	0, 5	3, 3

Nash equilibria

profile $S = (a_1 \ldots a_n)$

$S_i = (a_1 \ldots \boxed{a_{i-1}\ a_{i+1}} \ldots a_n)$ given a player i

excluding a_i

$S = (s_i, a_i)$

a strategy $a^* \in A_i$ is player i's best response to S_i if:

for any $a \in A_i$. $u_i(S_i, a) \leq u_i(S_i, a^*)$

"all other's action" $(a_1 \ldots a_{i-1}\ a_{i+1} \ldots a_n)$

A profile $(a_1 \ldots a_n)$ is a Nash equilibrium if for each player i, a_i is best response to S_i

Prisoner's dilemma:

for player 1, $S_1 = a_2$.

if $S_1 = C$ $(a_2 = C)$

$u(C, C) = 1$. if $a_1 = C$. $\Rightarrow$ C is best response to S_1 when $S_1 = C$.

$u(C, D) = 0$ if $a_1 = D$.

if $S_1 = D$ $(a_2 = D)$

$u(D, C) = 5$ if $a_1 = C$. $\Rightarrow$ C is best response to S_1 when $S_1 = D$

$u(D, D) = 3$ if $a_1 = D$

for player 2. $S_2 = a_1$

if $S_2 = C$.

$\begin{cases} u(C, C) = 1 \\ u(C, D) = 0 \end{cases} \Rightarrow$ C is best response.

if $S_2 = D$

$\begin{cases} u(D, C) = 5 \\ u(D, D) = 3 \end{cases} \Rightarrow$ C is best response

$\therefore$ Nash equilibrium: (C, C), utility at NE: $(1, 1)$

Coordination Games

We can choose drive left or right

if choose the same ⇒ safe.

if not ⇒ collision.

Suppose safe → 1 collision → −1

Formulation:

$N = (1, 2)$

$A = (A_1, A_2)$ $A_i = (L, R)$ ⇒

$u = (u_1, u_2)$

	L	R
L	1, 1	−1, −1
R	−1, −1	1, 1

Nash Equilibrium

For player 1 $S_1 = a_2$.

if $S_1 = L$. $u(S_1, L) > u(S_1, R) \Rightarrow$ L is best response.

if $S_1 = R$. $u(S_1, R) > u(S_1, L) \Rightarrow$ R is best response.

For player 2 $S_2 = a_1$.

if $S_2 = L$. $u(S_2, L) > u(S_2, R) \Rightarrow$ L is best response.

if $S_2 = R$. $u(S_2, R) > u(S_2, L) \Rightarrow$ R is best response.

$\therefore$ Nash equilibrium: (L, L) or (R, R) utility: $(1, 1)$

The matching Pennies Game

We each flip a coin.

- if match → I win
- if not match → you win

(win → 1, lose → −1)

Formulation.

$N = (1, 2)$, $A = (A_1, A_2)$, where $A_i = (H, T)$

$u = (u_1, u_2)$ ⇒

	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

Nash equilibrium:

For player 1 $S_1 = a_2$.

if $S_1 = H$. $u(S_1, H) > u(S_1, T) \Rightarrow$ H is best response.

if $S_1 = T$. $u(S_1, T) > u(S_1, H) \Rightarrow$ T is best response.

For player 2 $S_2 = a_1$.

if $S_2 = H$. $u(S_2, T) > u(S_2, H) \Rightarrow$ T is best response.

if $S_2 = T$. $u(S_2, H) > u(S_2, T) \Rightarrow$ H is best response.

$\therefore$ Nash equilibrium: No Nash Equilibrium!

⇒ Zero-Sum Game

in Game ($N = (1, 2)$ $A = (A, B)$ $u = (u_1, u_2)$)

"For all $a \in A$, $b \in B$, $u_1(a, b) + u_2(a, b) = 0$"

⇒ What is good for one player is bad for another

Unique NE payoff: if more than one NE exist (say $(a_1, b_1), (a_2, b_2)$) than $u_i(a_1, b_1) = u_i(a_2, b_2)$

in Zero-Sum Game:

if (a, b) is a NE, then for any $x \in A$:

$$\min_{y \in B} u_1(a, y) = u_1(a, b) \geq \min_{y \in B} u_1(x, y).$$

Proof:

First we prove $\min_{y \in B} u_1(a, y) = u_1(a, b)$

Suppose $b' = \arg\min_{y \in B} u_1(a, y) \Rightarrow u_1(a, b') = \min_{y \in B} u_1(a, y)$.

$u_2(a, b') = \max_{y \in B'} u_2(a, y)$

$\Rightarrow u_2(a, b') = u_2(a, b)$ (Since NE: b is best response to $S_2 = a$.)

$\therefore u_1(a, b') = u_1(a, b) = \min_{y \in B} u_1(a, y)$

Then. We prove $u_1(a, b) \geq \min_{y \in B} u_1(x, y)$.

Assume there exist $a' \in A$. $u_1(a, b) < \min_{y \in B} u_1(a', y) \leq u_1(a', b)$.

$u_1(a, b) \leq u_1(a', b)$ contradicts to NE ($a = \arg\max_{x \in A} u_1(x, b)$)

$\therefore$ Contradiction! $u_1(a, b) \geq \min_{y \in B} u_1(x, y)$.

Observation of Zero-Sum Game

- $u_1(a, b)$ is maximum among the minimum of $u_1(x, y)$
- $u_2(a, b)$ is maximum among the minimum of $u_2(x, y)$

"Mini-max"

Mixed Strategy

a lottery of its actions: $p_i: A_i \rightarrow [0, 1]$

$\sum_{a \in A} p_i(a) = 1$

$$u_i(p_1 \ldots p_n) = \sum_{(a_1 \ldots a_n) \in A} \underbrace{u_i(a_1 \ldots a_n)}_{\text{utility of a profile}} \underbrace{\prod_{k=1}^{n} p_k(a_k)}_{\text{probability of that profile}}$$

"Expectation"

mixed Nash Equilibrium:

if no one can deviate by a better mixed strategy